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II. Walras and Nontâtonnement Equilibria

Contents

FOREWORD

We shall present two **decentralized models** of general equilibrium, the Walras one and a slight modification of Walras's model, a nontâtonnement model. With the Walras model, we find an equilibrium price, which is an equilibrium for the tâtonnement process. The behavior of consumers is described by **demand functions or demand maps**: Knowing the equilibrium price, the choice of a consumer is given through the demand function or the demand map. The problem is then to find a price which provides an allocation of a scarce resource.

We shall begin with a simple version of the 1954 Arrow-Debreu Theorem, a prototype of many of the theorems stating the existence of an equilibrium. This theorem is easily derived from a very versatile tool, the Ky Fan inequality. This result is equivalent to the 1910 Brouwer Fixed Point Theorem, the corner stone of Nonlinear Analysis.

But using the Brouwer Fixed Point Theorem amounts to transforming the problem under investigation to a fixed point problem, for which one can apply this fixed point theorem or its numerous variations or extensions. The "fixed point" format being quite rigid, one encounters the risk of more or less considerable loss of information by doing so. It also often happens that this transformation may require additional assumptions and useless technical difficulties.

This is the reason why so many statements logically equivalent to the Brouwer Fixed Point Theorem — constituting the corpus of nonlinear analysis — have been designed to be readily adapted to classes of specific problems.

Among these equivalent results, Ky Fan Inequality incorporates, in some sense, more **labor-value**[1], and therefore, might be more useful.

We thus prove that Ky Fan's Inequality is equivalent to the Brouwer Fixed Point Theorem in the second section, and use it to derive the Arrow-Debreu Theorem for both the simple Walras allocation model and for general excess demand maps.

[1]Although two statements $\mathcal{P}$ and $\mathcal{Q}$ may well be equivalent, it is common experience that the proof of one of the implications, say $\mathcal{P} \Longrightarrow \mathcal{Q}$, is more difficult or involves deeper results than the proof of the converse. In this case, one can say that $\mathcal{Q}$ incorporates more labor value than $\mathcal{P}$ and thus, expect as a general rule that the statement $\mathcal{Q}$ may be more useful than $\mathcal{P}$.

By the way, the most pleasant or intuitive statements are quite often the ones with the least labor value.

This Arrow-Debreu theorem asserts that contrary to a primary intuition, a deep mathematical metaphor, in a extremely simplified framework, justifies the functioning of a decentralized mechanism. Meanwhile, it has been improved in many ways, naturally, and adapted to more and more complicated situations. We shall concentrate here only on the basic features.

Yet, this model is quite imperfect, and we have already mentioned some flaws: This a static model, and its dynamical version is not viable. In the second decentralized equilibrium model, the nontâtonnement model, we shall look for an equilibrium allocation, which is an equilibrium for a **dynamical system** describing the dynamical behavior of consumers. Demand functions are replaced by **change functions**, which described how fast a consumer changes her consumption knowing it and the price. An equilibrium is composed of an allocation and a price that no consumer will change (this describes the satisfaction of the consumers).

For proving the existence of such an equilibrium, we shall deduce from Ky Fan's Inequality a General Equilibrium Theorem, providing an equilibrium $\bar{x}$ of a set-valued map F (i.e., a solution to the inclusion $0 \in F(\bar{x})$ satisfying the constraints described by $\bar{x} \in K$.) Besides standard mild conditions on F, we shall assume that K is a **convex compact viability domain of** F: the latter means that we can find at every point x of the compact convex subset K an element $v \in F(x)$ tangent to K at x.

We start with an allocation set K defined

$$K := \left\{ x := (x_1, \ldots, x_n) \in \prod_{i=1}^{n} L_i \;\middle|\; \sum_{i=1}^{n} x_i \in M \right\} \qquad (0.1)$$

where $M \subset Y$ is a subset of scarce resources of the commodity space $Y := \mathbf{R}^l$ and where L_i are the consumption sets of the n consumers $i = 1, \ldots, n$.

We assume that the prices range over the **price simplex**

$$S^l := \left\{ p \in \mathbf{R}^l_+ \mid \sum_{h=1}^{l} p^h = 1 \right\}$$

1 The Walras Model

1.1 The Arrow-Debreu Theorem

We present a very short description of the Walras model, based on the representation of consumers by n **demand functions**

$$d_i : (p,r) \in S^l \times \mathbf{R} \rightarrow d_i(p,r) \in L_i$$

associating to a price p and an income r the i^{th} consumption of Mrs. i.

This a very crude description of the behavior of a consumer. Later on, we shall attenuate the "automatic feature" by accepting set-valued demand functions, that are called **demand maps**. As we have seen, some demand map can be derived from utility or loss functions.

Demand functions are required to obey a **budgetary rule** called the **Walras law**. It states that it is **forbidden to spend more than her income.**

This can be translated mathematically in the following form:

$$\forall\, p \in S^l, \ \langle p, d_i(p,r)\rangle \ \leq \ r$$

since $\langle p, d_i(r,p)\rangle$ is the amount of monetary units spent to acquire $d_i(p,r)$ when p is the price and r is the income.

When M is the given subset of resources, we regard its support function $\sigma_M(p) := \sup_{y\in M} < p, y >$ as the **total income.**

The second basic ingredient of the Walras model is the **allocation among the consumers of (part of) the total income**

$$\sum_{i=1}^{n} r_i(p) \ \leq \ \sigma_M(p)$$

derived from the set of available resources. (This is an assumption, and not a conclusion: the Walras model associates an equilibrium with any allocation of the income, fair or unfair).

Therefore, when a consumer is described by her demand function d_i and is supplied with an income function r_i, her decision depends upon the knowledge of the price: she associates with a price p the commodity $d_i(p, r_i(p))$.

The role of a decentralized mechanism is then to provide every consumer a common price, i.e., a message summarizing the unknown set of available resources and the behavior of all consumers, such that the resulting consumptions $d_i(p, r_i(p))$ do form an allocation, i.e., satisfy

$$0 \ \in \ \sum_{i=1}^{n} d_i(p, r_i(p)) - M \ =: \ E(p)$$

where $E(p)$ is the **excess demand**.

Theorem 1.1 (Arrow-Debreu) *Assume that the demand functions d_i of the n consumers are continuous and that the set of scarce resources is closed, convex, bounded above and satisfy $M = M - \mathbf{R}^l_+$.*

Assume also that the n income functions $r_i(p)$ are continuous and satisfy $\sum_{i=1}^n r_i(p) \leq \sigma_M(p)$.

If the Walras law is satisfied, then there exists at least a Walras equilibrium.

This theorem is the prototype of many theorems of general equilibrium theory. We shall derive it from Ky Fan's inequality:

1.2 The Ky Fan Inequality

We shall prove that the 1910 Brouwer Fixed Point Theorem[2]:

Theorem 1.2 (Brouwer Fixed Point Theorem) *Let K be a compact convex subset of a finite dimensional vector-space and $f : K \mapsto K$ be a continuous (single-valued) map. Then there exists a fixed point $f(\overline{x}) = \overline{x} \in K$ of f in K.*

is equivalent to the Ky Fan inequality:

Theorem 1.3 (Ky Fan Inequality) *Let K be a compact convex subset of a finite dimensional vector-space[3] X and $\varphi : X \times X \mapsto \mathbf{R}$ be a function satisfying*

$$\begin{cases} i) & \forall\, y \in K, \quad x \mapsto \varphi(x,y) \text{ is lower semicontinuous} \\ ii) & \forall\, x \in K, \quad y \mapsto \varphi(x,y) \text{ is concave} \\ iii) & \forall\, y \in K, \quad \varphi(y,y) \leq 0 \end{cases} \tag{1.1}$$

Then, there exists $\overline{x} \in K$, a solution to

$$\forall\, y \in K, \ \varphi(\overline{x}, y) \leq 0 \tag{1.2}$$

[2]See [1, Appendix B] for a proof of the Brouwer Fixed Point Theorem based on Sperner's Lemma and [2, Chapter II] for a proof based on differential geometry.

[3]Actually, this theorem holds true for any Hausdorff locally convex topological vector space and in particular, for weak topologies of Banach spaces X.

Proof

— **Proof of Ky Fan Inequality** We shall derive a contradiction from the negation of the conclusion:

$$\forall\, x \in K,\ \exists\, y \in K \text{ such that } \varphi(x,y) > 0$$

so that K can be covered by the subsets

$$\mathcal{V}_y := \{x \in K \mid \varphi(x,y) > 0\}$$

which are open by assumption (1.1) i). Since K is compact, it can be covered by n such open subsets $\mathcal{V}_{y_i}$. Let us consider a continuous partition of unity[4] $(\alpha_i)_{i=1,\ldots,n}$ associated with this open covering of K and define the map $f : K \mapsto X$ by

$$\forall\, x \in K,\ f(x) := \sum_{i=1}^{n} \alpha_i(x) y_i$$

It maps K to itself because K is convex and the elements y_i belong to K. It is also continuous, so that Brouwer's Fixed Point Theorem implies the existence of a fixed point $\overline{y} = f(\overline{y}) \in K$ of f. Assumption (1.1) ii) imply that

$$\varphi(\overline{y},\overline{y}) = \varphi(\overline{y}, \sum_{i=1}^{n} \alpha_i(\overline{y}) y_i) \geq \sum_{i=1}^{n} \alpha_i(\overline{y}) \varphi(\overline{y}, y_i)$$

Let us introduce

$$I(\overline{y}) := \{i = 1, \ldots, n \mid \alpha_i(\overline{y}) > 0\}$$

It is not empty because $\sum_{i=1}^{n} \alpha_i(\overline{y}) = 1$. Furthermore

$$\sum_{i=1}^{n} \alpha_i(\overline{y}) \varphi(\overline{y}, y_i) = \sum_{i \in I(\overline{y})} \alpha_i(\overline{y}) \varphi(\overline{y}, y_i) > 0$$

because, whenever i belongs to $I(\overline{y})$, $\alpha_i(\overline{y}) > 0$, so that $\overline{y}$ belongs to $\mathcal{V}_{y_i}$, and thus, by the very definition of this subset, $\varphi(\overline{y}, y_i) > 0$. Hence, we have

[4] A continuous partition of unity associated with a covering of K by n open subsets $\mathcal{V}_i$ is a sequence of n continuous maps $\alpha_i : K \mapsto \mathbf{R}$ such that,

$$\forall\, x \in K,\ \sum_{i=1}^{n} \alpha_i(x) = 1,\ \forall\, i = 1, \ldots, n,\ \alpha_i(x) \geq 0 \ \&\ \text{support}(\alpha_i) \subset \mathcal{V}_i$$

Such continuous partitions of unity do exist when K is a compact metric space.

proved that $\varphi(\overline{y}, \overline{y})$ is strictly positive, a contradiction of assumption (1.1) *iii*).

— **Proof of Brouwer's Theorem** We associate with the continuous map $f : K \mapsto K$ the function φ defined on $K \times K$ by

$$\varphi(x, y) := \langle f(x) - x, y - x \rangle$$

which satisfies obviously the assumptions (1.1) of Ky Fan Inequality. Then there exists $\overline{x} \in K$ such that inequality (1.2) holds true. By taking $y := f(\overline{x}) \in K$, we infer that $\|f(\overline{x}) - \overline{x}\| \leq 0$, and thus, that $\overline{x}$ is a fixed point of f. □

Remark — Minimax and Lop-sided Minimax Inequalities
The Ky Fan inequality implies readily the von Neumann Minimax Theorem:

Theorem 1.4 (Minimax) *Let X and Y be Banach spaces*[5]*, $L \subset X$ and $M \subset Y$ be compact convex subsets and $f : L \times M \mapsto \mathbf{R}$ be a real valued function satisfying*

$$\begin{cases} i) & \forall\, y \in M, \;\; x \mapsto f(x, y) \text{ is lower semicontinuous and convex} \\ ii) & \forall\, x \in L, \;\; y \mapsto f(x, y) \text{ is upper semicontinuous and concave} \end{cases}$$

Then there exists a saddle point $(\bar{x}, \bar{y}) \in L \times M$ *of f:*

$$\forall\, (x, y) \in L \times M, \;\; f(\bar{x}, y) \;\leq\; f(\bar{x}, \bar{y}) \;\leq\; f(x, \bar{y})$$

Proof — We apply the Ky Fan Inequality with $K := L \times M$ and φ defined by

$$\varphi((\bar{x}, \bar{y}), (x, y)) := f(\bar{x}, y) - f(x, \bar{y}) \;\; \square$$

Actually, we often need a weaker version of the Minimax Theorem, called the Lop-Sided Minimax Theorem, relying only on the Searation Theorem. We recall its statement:

Theorem 1.5 (Lop-Sided Minimax Theorem) *Let X and Y be finite dimensional vector-space*[6]*, $L \subset X$ be a compact convex subset, $M \subset Y$ be a convex subset and $f : L \times M \mapsto \mathbf{R}$ be a real valued function satisfying*

$$\begin{cases} i) & \forall\, y \in M, \;\; x \mapsto f(x, y) \text{ is lower semicontinuous and convex} \\ ii) & \forall\, x \in L, \;\; y \mapsto f(x, y) \text{ is and concave} \end{cases}$$

[5]actually, Hausdorff locally convex topological vector spaces.

[6]or, more generally, a Banach space or even, a Hausdorff locally convex topological vector spaces.

Then there exists $\bar{x} \in L$ satisfying

$$\forall\, y \in M,\quad f(\bar{x}, y) \;\le\; \inf_{x \in L} \sup_{y \in M} f(x, y) \;=\; \sup_{y \in M} \inf_{x \in L} f(x, y)$$

Proof — We refer to Theorem 6.2.7. of APPLIED NONLINEAR ANALYSIS for an instance of proof using only the Separation Theorem. □

1.3 Proof of Arrow-Debreu's Theorem

This theorem is a simple consequence of the above Ky Fan's Inequality (Theorem 1.3) which we shall prove just after this theorem.

We apply it to the following function φ defined on $S^l \times S^l$ by

$$\varphi(p, q) \;:=\; \sum_{i=1}^{n} (\langle q, d_i(p, r_i(p))\rangle - r_i(q))$$

It is obviously continuous with respect to p and concave with respect to q. Walras law implies that

$$\varphi(q, q) \;=\; \sum_{i=1}^{n} (\langle q, d_i(q, r_i(q))\rangle - r_i(q)) \;\le\; 0$$

Hence, the assumptions of Ky Fan's Inequality Theorem are satisfied, so that there exists a price $\bar{p} \in S^l$ such that

$$\forall\, q \in S^l,\;\; \varphi(\bar{p}, q) = \sum_{i=1}^{n} \langle q, d_i(\bar{p}, r_i(\bar{p}))\rangle - \sigma_M(q) \;\le\; 0$$

The above inequality is still true for $q \in \mathbf{R}^n_l$, and even for any $q \in \mathbf{R}^l$ because, since we have assumed that $M = M - \mathbf{R}^l_+$, we know that

$$\sigma_{M-\mathbf{R}^l_+}(p) \;=\; \begin{cases} \sigma_M(p) & \text{if } \;p \in \mathbf{R}^l_+ \\ +\infty & \text{if } \;p \notin \mathbf{R}^l_+ \end{cases}$$

since $\left(-\mathbf{R}^l_+\right)^- = \mathbf{R}^l_+$.

Hence we derive from the Separation Theorem that

$$\sum_{i=1}^{n} d_i(\bar{p}, r_i(\bar{p})) \in \overline{co}(M - \mathbf{R}^l_+) = M \;\;\square$$

Remark — Collective Walras Law

The advantage of the Walras law is that it is a decentralized condition. If each consumer conforms to it, then an equilibrium does exist thanks to the above theorem.

But the proof shows that it is enough to assume that the weaker — but not decentralized — collective Walras law

$$\sum_{i=1}^{n}\langle p, d_i(p,r_i)\rangle \leq \sum_{i=1}^{n} r_i$$

is satisfied. This weaker version admits monetary transactions among consumers to modify their initial income functions. □

1.4 Walras Equilibria of Excess Demand Maps

We started with the simplest model, but the proof of the Arrow-Debreu Theorem from the Ky Fan Inequality shows that this theorem can be readily extended to the case when

- demand functions $d_i(p,r) : S^l \times \mathbf{R} \mapsto L_i$ are replaced by **demand maps** $D_i(p,r) : S^l \times \mathbf{R} \leadsto L_i$
- a **supply map** $S_M : S^l \leadsto M$ is introduced (instead of the constant supply map $S_M(p) \equiv p$)

We have seen examples of such demand and a supply maps in the optimal allocation model.

We recall the definition of **support function** σ_K and **lower support function** $\sigma_K^\flat$ of a subset $K \subset X$:

$$\forall\, p \in X^\star,\;\; \sigma_K(p) := \sup_{x\in K}\langle p,x\rangle \;\;\&\;\; \sigma_K^\flat(p) := \inf_{x\in K}\langle p,x\rangle \;=\; -\sigma_K(-p)$$

and that the Separation Theorem can also be written in the form:

$$\overline{co}(K) \;=\; \left\{x \,\in\, X \mid \forall\, p \,\in\, X^\star,\;\; \langle p,x\rangle \,\geq\, \sigma_K^\flat(p)\right\}$$

We associate with the supply map its **income function** r defined by

$$\forall\, q \in S^l,\;\; r(q) \;:=\; \sigma_{S_M(q)}(q) \;\leq\; \sigma_{M(q)}(p)$$

We shall assume that this income function is allocated among consumers by providing them income functions r_i satisfying

$$\forall\, p \in S^l,\;\; \sum_{i=1}^{n} r_i(p) \;\leq\; r(p)$$

Describing the behavior of economic agents by these supply and demand map, we define the **excess demand map** E by

$$\forall\, p \in S^l, \;\; E(p) \;:=\; \sum_{i=1}^{n} D_i(p, r_i)(p) - S_M(p) - \mathbf{R}^l_+$$

The **tâtonnement process** is described by the following differential inclusion

$$p'(t) \;\in\; E(p(t))$$

in the continuous version and

$$p_{n+1} - p_n \;\in\; \delta_n E(p_n)$$

in the discrete version to describe the **Law of Supply and Demand**: the price increases when the demand increases.

Hence, the **Walras equilibrium** is a price $\overline{p}$ which clears the market in the sense that

$$0 \;\in\; E(\overline{p})$$

In order to prove the existence of such a Walras equilibrium, we need some continuity requirement. Without introducing right away the whole machinery concerning continuity issues of set-valued maps[7], we just define **hemicontinuity** which is sufficient for our immediate purpose.

Definition 1.6 (Upper Hemicontinuous Map) *We shall say that a set-valued map $F : X \rightsquigarrow Y$ is* **upper hemicontinuous** *at $x_0 \in \mathrm{Dom}(F)$ if and only if for any $p \in Y^\star$, the function $x \mapsto \sigma_{F(x)}(p)$ is upper semicontinuous at x_0 (or the function $x \mapsto \sigma^\flat_{F(x)}(p)$ is lower semicontinuous). It is said to be upper hemicontinuous if and only if it is upper hemicontinuous at every point of* $\mathrm{Dom}(F)$.

Theorem 1.7 *Let us assume that an excess demand map $E : S^l \rightsquigarrow Y$ is upper hemicontinuous and that its images are closed, convex, bounded above and satisfy*

$$\forall\, p \in S^l, \;\; E(p) \;=\; E(p) - \mathbf{R}^l_+$$

Assume also that it satisfies the Walras law

$$\forall\, q \in S^l, \;\; \sigma^\flat_{E(q)}(q) \;\leq\; 0$$

Then there exists at least a Walras equilibrium $\overline{p} \in S^l$, a solution to

$$0 \;\in\; E(\overline{p})$$

[7] for which we refer to SET-VALUED ANALYSIS, [3, Aubin & Frankowska]

Proof — It is the same than the proof of Arrow-Debreu's Theorem above. We apply Ky Fan's Inequality (Theorem 1.3) to the following function φ defined on $S^l \times S^l$ by

$$\varphi(p,q) := \sum_{i=1}^{n} \sigma^{\flat}_{E(p)}(q)$$

It is obviously finite (because $E(p)$ is bounded above) and concave with respect to q. Since the excess demand map is assumed to be upper hemicontinous, we infer that φ is lower semicontinuous with respect to p.

Walras law implies that

$$\varphi(q,q) = \sum_{i=1}^{n} \sigma^{\flat}_{E(q)}(q) \leq 0$$

Hence, the assumptions of Ky Fan's Inequality Theorem are satisfied, so that there exists a price $\bar{p} \in S^l$ such that

$$\forall\, q \in S^l, \;\; \varphi(\bar{p},q) = \sum_{i=1}^{n} \sigma^{\flat}_{E(\overline{p})}(q) \leq 0$$

The above inequality is still true for $q \in \mathbf{R}^l_+$.

We thus deduce that for any

$$\forall\, q \in \mathbf{R}^l, \;\; \varphi(\bar{p},q) = \sum_{i=1}^{n} \sigma^{\flat}_{E(\overline{p})-\mathbf{R}^l_+}(q) \leq 0$$

Hence we derive from the Separation Theorem that 0 belongs to the closed convex set $E(\overline{p})$ since this set is closed and convex. □

Consider now the case when **excess demand map** E is derived from demand and supply maps:

$$\forall\, p \in S^l, \;\; E(p) := \sum_{i=1}^{n} D_i(p,r_i)(p) - S_M(p) - \mathbf{R}^l_+$$

We shall assume that the values $D_i(p,r)$ and $S_M(p)$ of the demand and supply functions are closed and convex. Therefore, they can be characterized by their support functions:

$$D_i(p,r) = \left\{x \mid \forall\, q \in Y^{\star}, \; \sigma^{\flat}_{D_i(p,r)}(q) \leq \langle q,x\rangle\right\}$$

and

$$S_M(p) \; = \; \left\{ y \mid \forall\, q \in Y^\star,\ \langle q, y\rangle \leq \sigma_{S_M(p)}(q) \right\}$$

We shall assume also that the demand maps satisfy the **Walras law** which can be written in the form

$$\forall\, q \in S^l,\ \ \sigma^\flat_{D_i(q,r)}(q) \; \leq \; r$$

or the **collective Walras law**

$$\forall\, q \in S^l,\ \ \sum_{i=1}^n \sigma^\flat_{D_i(q,r_i)}(q) \; \leq \; \sum_{i=1}^n r_i \tag{1.3}$$

We derive the following consequence:

Theorem 1.8 *Let us assume that the demand and supply maps are upper hemicontinuous, that the values $D_i(p,r)$ are convex and compact and that the values $S_M(p)$ of the supply demand are closed, convex, bounded above and satisfy*

$$\forall\, p \in S^l,\ \ S_M(p) \; = \; S_M(p) - \mathbf{R}^l_+$$

Assume also that the n income functions $r_i(p)$ are continuous and satisfy $\sum_{i=1}^n r_i(p) \leq \sigma_{S_M(p)}(p)$.

If the collective Walras law (1.3) is satisfied, then there exists at least a Walras equilibrium $\overline{p} \in S^l$, a solution to

$$0 \; \in \; \sum_{i=1}^n D_i(\overline{p}, r_i)(\overline{p}) - S_M(\overline{p})$$

Proof — Since the images $D_i(p,r)$ are convex and compact and the images $S_M(p)$ are closed and convex, we infer that the images

$$E(p) \; := \; \sum_{i=1}^n D_i(p, r_i(p)) - S_M(p) \; = \; E(p) - \mathbf{R}^l_+$$

are closed, convex and bounded above.

Since

$$\sigma^\flat_{E(p)}(q) \; := \; \sum_{i=1}^n \sigma^\flat_{D_i(p,r_i(p))}(q) - \sigma_{(S_M(p)}(q)$$

we infer that $E(\cdot)$ is upper hemicontinuous. Finally, it satisfies

$$\sigma^\flat_{E(q)}(q) \; = \; \sum_{i=1}^n \sigma^\flat_{D_i(p,r_i(p)}(q) - \sigma_{S_M(q)}(q) \; \leq \; \sum_{i=1}^n r_i(q) - r(q) \; \leq \; 0$$

Hence the assumptions of Theorem 1.7 are satisfied and the conclusion ensues. □

1.5 Notes on Upper Hemicontinuous Maps

Proposition 1.9 *The graph of an upper hemicontinuous set-valued map with closed convex values is closed.*

Conversely, if the graph of F is closed and if the images of F remain in a compact set, then F is upper hemicontinuous.

Proof

a) Consider a sequence of elements (x_n, y_n) of Graph(F) converging to a pair (x, y). Then, for every $p \in Y^\star$,

$$\langle p, y\rangle \; = \; \lim_{n\to\infty} \langle p, y_n\rangle \leq \limsup_{n\to\infty} \sigma_{F(x_n)}(p) \; \leq \; \sigma_{F(x)}(p)$$

by the upper semicontinuity of $x \mapsto \sigma_{F(x)}(p)$. This inequality implies that $y \in F(x)$ since these subsets are closed and convex, thanks to the Separation Theorem.

We thus have shown that (x, y) belongs to Graph(F).

b) To show that F is upper hemicontinuous, take any $\lambda \in \mathbf{R}$ and show that the upper sections

$$\left\{x \mid \lambda \; \leq \; \sigma_{F(x)}(q)\right\}$$

are closed. Let a sequence of elements x_n of such an upper section converge to x. Since the values $F(x_n)$ are compact, there exist $y_n \in F(x_n)$ such that $\langle q, y_n\rangle = \sigma_{F(x_n)}(q) \geq \lambda$. But the sequence y_n residing in a compact set, a subsequence (again denoted by) y_n converges to some y, which belongs to $F(x)$ since the graph of F is closed. Hence $\lambda \leq \langle q, y\rangle \leq \sigma_{F(x)}(q)$, and the second statement of the Lemma ensues. □

2 The Nontâtonnement Model

2.1 Dynamical Allocation of Scarce Resources

Instead of describing the decentralized behavior of a consumer by a Walras demand function, which makes sense in the static case, we shall capture the essence of a demand function to take into account the evolutionary aspect by transferring it to a "change function"

$$(x, p) \; \mapsto \; c_i(x, p)$$

associating with each commodity x owned by consumer i and the price p she observes on the market the velocity with which she will change her commodity

The idea is to describe a **dynamical behavior** of consumer i by the differential equation

$$x_i'(t) \; = \; c_i(x_i(t), p(t))$$

It is decentralized in the sense that the decision of consumer i does not involve the knowledge of the set M of available resources nor the behavior of her fellow consumers.

Her decision depends only upon her current consumption $x_i(t)$ and the "current price" (also called "spot price") $p(t)$ at time t: at each time t, she decides to increase, decrease or maintain constant her consumption according to her consumption and to the price she observes.

Now, we have to introduce an a priori law for price behavior. In the simplest case, we can choose prices in the price simplex

$$S^l \; := \; \{ p \in \mathbf{R}_+^l \mid \sum_{h=1}^{l} p^h = 1 \}$$

But we can take into consideration external laws or regulations, and for that purpose, introduce a set-valued map $P : K \rightsquigarrow \mathbf{R}_+^l$, called a **pricing map**, associating to each allocation x a subset $P(x) \subset S^l$ of feasible prices (allowed by external regulations, for instance).

Hence, the prices are requested to obey the evolution law:

$$\forall \, t \geq 0, \;\; p(t) \; \in \; P(x(t))$$

By summarizing, the dynamics of the evolution of the consumption is described by

$$\forall \, t \geq 0, \;\; \begin{cases} i) & x_i'(t) \; = \; c_i(x_i(t), p(t)) \;\; (i = 1, \ldots, n) \\ & \\ ii) & p(t) \; \in \; P(x(t)) \end{cases}$$

Equilibria of this dynamical system are solutions $(\bar{x}, \bar{p})$ to the system

$$\begin{cases} i) & c_i(\bar{x}_i, \bar{p}) \; = \; 0 \;\; (i = 1, \ldots, n) \\ ii) & \bar{p} \; \in \; P(\bar{x}) \end{cases} \tag{2.1}$$

We first address the problem of finding viable equilibria, i.e., equilibria satisfying

$$\bar{x}_i \in L_i \ (i=1,\ldots,n) \ \& \ \sum_{i-1}^{n} \bar{x}_i \in M$$

2.2 The Regulation Map

We recall that the **tangent cone** $T_K(x)$ to a convex subset K at $x \in K$ is the closed cone spanned by $K - x$, which is convex:

$$T_K(x) = \overline{\bigcup_{h>0} \frac{K-x}{h}}$$

and the **normal cone** to K at x defined by

$$N_K(x) := (T_K(x))^- := \left\{p \in \mathbf{R}^{l^\star} \mid \langle p, x\rangle = \sigma_K(p)\right\}$$

We need the concept of tangent cone to associate with the behavior of the consumers described by change functions and the set M of scarce resources the **regulation map** Π_M defined by

$$\forall\, x \in K,\ \Pi_M(x) := \left\{ p \in P(x) \mid \sum_{i=1}^{n} c_i(x_i, p) \in T_M\left(\sum_{i=1}^{n} x_i\right) \right\}$$

This regulation map relates each allocation $x \in K$ to prices such that the associated total change of consumptions is tangent to the set of scarce resources; The objective is to bring back the total consumption inside the set of scarce resources.

We posit now the assumptions we need to prove our equilibrium theorem.
— ASSUMPTIONS ON THE CONSUMPTION AND RESOURCE SETS:

$$\begin{cases} i) \quad M = M - \mathbf{R}^l_+ \text{ is a closed convex subset} \\ ii) \quad \forall\, i = 1,\ldots,n,\ L_i = L_i + \mathbf{R}^l_+ \text{ is closed and convex} \\ iii) \quad 0 \in \operatorname{Int}\left(\sum_{i=1}^{n} L_i - M\right) \\ iv) \quad M \subset \underline{y} - \mathbf{R}^l_+ \ \& \ \forall\, i = 1,\ldots,n,\ L_i \subset \underline{x}_i + \mathbf{R}^l_+ \end{cases} \tag{2.2}$$

These are the standard assumptions, the most severe one being the convexity of the consumption and resource sets, which is needed to gaurantee the existence of an equilibrium (but not needed for the existence of viable evolution of allocations).

Recall that implies that under these assumptions, the tangent cone to the set of allocations is derived from the tangent cones to the consumption and resource sets by the formula[8]:

$$T_K(x) := \left\{ v := (v_1, \ldots, v_n) \in \prod_{i=1}^{n} T_{L_i}(x) \;\middle|\; \sum_{i=1}^{n} v_i \in T_M\left(\sum_{i=1}^{n} x_i\right) \right\} \tag{2.3}$$

— ASSUMPTIONS ON THE CHANGE FUNCTIONS:

$$\left\{ \begin{array}{ll} i) & c_i(x,p) := c_i(x) + G_i(x)p \text{ is affine, where} \\ ii) & c_i : L_i \mapsto Y \text{ is continuous} \\ iii) & G_i : L_i \mapsto \mathcal{L}(Y^\star, Y) \text{ is continuous} \\ iv) & \forall\, x_i \in L_i,\ p \in \operatorname{Im}(P),\ c_i(x_i, p) \in T_{L_i}(x_i) \end{array} \right. \tag{2.4}$$

— ASSUMPTIONS ON THE FEEDBACK MAP:

$$\left\{ \begin{array}{ll} i) & \text{Graph}(P) \text{ is closed and the images of } P \text{ are convex} \\ ii) & \forall\, x \in K,\ N_M\left(\sum_{i=1}^{n} x_i\right) \cap S^l \subset P(x) \end{array} \right. \tag{2.5}$$

Constant pricing maps $P(x) \equiv S^l$ satisfy naturally these assumptions. Assumption $M = M - \mathbf{R}^l_+$ implies that

$$N_M\left(\sum_{i=1}^{n} x_i\right) \subset \mathbf{R}^l_+$$

These assumptions are just technical ones. We shall need other "consistency" assumptionsm which play a very deep role, and which should have the same degree of economic interpretation than the conclusion of the theorem. They take the form of budgetary rules.

[8]This formula remains true when the consumption and resource sets are assumed to be sleek instead of convex.

2.3 The Instantaneous Walras Law

In order to guarantee the nonemptiness of the images of the regulation map Π_M **without knowing the set** M **of resources**, we appeal again to a mathematical implementation of the budgetary law enjoining consumers to spend less than they earn (in units of account).

In the case of our nontâtonnement model, it takes the form of the *collective instantaneous Walras law*:

$$\forall\, p \in S^l,\ \sum_{i=1}^{n} \langle p, c_i(x_i, p)\rangle \ \leq\ 0$$

This law itself can be decentralized further by requiring the change functions c_i to obey the (individual) **instantaneous Walras law**

$$\forall\, p \in S^l,\ \langle p, c_i(x_i, p)\rangle \ \leq\ 0$$

Indeed, we can portray this property by saying that it is forbidden to spend more monetary units than earned in continuous transactions. To be persuaded by this interpretation, it is enough to observe that the Walras law implies that along solutions to the system of differential equations (2.9), we have $\langle p(t), x_i'(t)\rangle \leq 0$, and thus, for all h small enough,

$$\langle p(t), x_i(t)\rangle \ \leq\ \langle p(t), x_i(t-h)\rangle + \varepsilon h$$

As we can see, the advantage of the instantaneous Walras law is that it does not depend upon the set M of scarce resources, as long as it satisfies assumptions (2.2)i).

2.4 The Nontâtonnement Equilibrium Theorem

Hence, the following Theorem is an existence result of an equilibrium of the nontâtonnement process which shares the same features than the Arrow-Debreu Theorem on the existence of a Walras equilibrium, an equilibrium of the tâtonnement process:

Theorem 2.1 *We posit the assumptions (2.2), (2.4) and (2.5). If the change functions c_i obey the collective instantaneous Walras law*

$$\forall\, p \in S^l,\ \sum_{i=1}^{n} \langle p, c_i(x_i, p)\rangle \ \leq\ 0$$

then there exists a viable equilibrium $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p}) \in K \times S^l$, *a solution to the system of equations:*

$$c_i(\bar{x}_i, \bar{p}) \;=\; 0 \quad (i = 1, \ldots, n)$$

Remark — When $P(\cdot)$ is no longer the constant map $P \equiv S^l$, we can assume that for all $x \in \prod_{i=1}^n L_i$, there exists a map $Q(x, \cdot) : S^l \mapsto P(x)$ satisfying the condition

$$\forall\, (x,p) \in \prod_{i=1}^n L_i \times S^l, \;\; \langle p, \sum_{i=1}^n c_i(x_i, Q(x,p)) \rangle \;\leq\; 0$$

Then the viability condition holds (2.7) true. □

We already mentionned that the collective instantaneous Walras law implies that the regulation map Π_M is nontirivial. This results from the following

Proposition 2.2 *We posit assumptions (2.2), (2.4) and*

$$\text{Graph}(P) \text{ is closed and the images of } P \text{ are convex} \qquad (2.6)$$

Then the regulation map $\Pi_M(\cdot)$ *is not trivial if and only if the "budgetary condition"*

$$\sup_{q \in N_M(\sum_{i=1}^n x_i)} \; \inf_{p \in P(x)} \left\langle q, \sum_{i=1}^n c_i(x_i, p) \right\rangle \;\leq\; 0$$

hods true.

Proof — The budgetary condition being obviously necessary, let us check that it is sufficient by using of the Separation Theorem. Indeed, assume that $\Pi_M(x) = \emptyset$ for some $x \in K$. This means that

$$0 \;\notin\; T_M\left(\sum_{i=1}^n x_i\right) - \left\{\sum_{i=1}^n c_i(x_i, p)\right\}_{p \in P(x)}$$

Since this subset is closed and convex, there exist $q \in Y^\star$ such that

$$\sup_{v \in T_M(\sum_{i=1}^n x_i),\; p \in P(x)} \left\langle q, v - \sum_{i=1}^n c_i(x_i, p) \right\rangle \;<\; 0$$

By fixing $p \in P(x)$, this implies first that

$$q \in N_M\left(\sum_{i=1}^n x_i\right) := \left(T_M\left(\sum_{i=1}^n x_i\right)\right)^-$$

By taking $v = 0 \in T_M\left(\sum_{i=1}^n x_i\right)$, we infer that

$$\sup_{p\in P(x)}\left\langle q, -\sum_{i=1}^n c_i(x_i,p)\right\rangle < 0$$

and therefore, that

$$\inf_{q\in N_M(\sum_{i=1}^n x_i)}\sup_{p\in P(x)}\left\langle q, -\sum_{i=1}^n c_i(x_i,p)\right\rangle < 0$$

a contradiction of the assumption. □

We recall that the condition $M = M - \mathbf{R}^l_+$ implies that

$$N_M\left(\sum_{i=1}^n x_i\right) \subset \mathbf{R}^l_+$$

so that the assumption of Proposition 2.2 implies the above characterization of the nonemptiness of $\Pi_M(x)$.

Theorem 2.1 on the existence of a nontâtonnement equilibrium then follows from Proposition 2.2 and the "nondecentralized" (but more general) Theorem2.3 below:

Theorem 2.3 *We posit assumptions (2.2), (2.4) and (2.6). If the* regulation map $\Pi_M(\cdot)$ *is nontirivial:*

$$\forall\, x \in K,\ \ \Pi_M(x) \neq \emptyset \tag{2.7}$$

then there exists a viable equilibrium $(\bar{x}_1,\ldots,\bar{x}_n,\bar{p}) \in K \times S^l$.

This Theorem follows from the Equilibrium Theorem 3.1 we shall prove in next subsection.

But before proving it, we emphasize the fact that the existence of an equilibrium is derived from the nonemptiness of the images of the regulation map Π_M, which itself is derived from the collective Walras law.

2.5 Towards an Abstract Equilibrium Model

How can we prove Theorem 2.3 (and thus, Theorem 2.1) ?

First, we introduce the map $c : X \times S^l \mapsto X$ defined by

$$c(x,p) := (c_1(x_1,p),\ldots,c_n(x_n,p))$$

and we set

$$K := \left\{ x := (x_1,\ldots,x_n) \in \prod_{i=1}^{n} L_i \;\middle|\; \sum_{i=1}^{n} x_i \in M \right\}$$

so that an equilibrium is a solution $(\overline{x},\overline{p}) \in K \times S^l$ to the system

$$c(\overline{x},\overline{p}) = 0 \;\&\; \overline{p} \in P(\overline{x})$$

We can simplify further the formulation of this problem by introducing the set-valued map $F : K \leadsto X$ defined by

$$F(x) := c(x,P(x)) = \{c(x,p)\}_{p\in P(x)} \tag{2.8}$$

Therefore, there exists an equilibrium $(\overline{x},\overline{p})$ if and only if $\overline{x}$ is an equilibrium of F in the sense that

$$\overline{x} \in K \;\&\; 0 \in F(\overline{x})$$

Hence, we have to appeal to existence of equilibria of set-valued maps.

But before, let us check what are the properties of the set-valued map F.

For that purpose, it is convenient to introduce the following definition:

Definition 2.4 *A convex subset $K \subset \mathrm{Dom}(F)$ satisfying property*

$$\forall\, x \in K,\;\; F(x) \cap T_K(x) \neq \emptyset$$

is said to be a viability domain *of F.*

This means that for any point $x \in K$, there exists at least a direction $v \in F(x)$ which is tangent to K at x[9].

[9]We shall see that the Viability Theorem states that K is viability domain of F if and only if that from any initial state $x_0 \in K$ starts at least one solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ which is viable in K in the sense that $x(t) \in K$ for any $t \geq 0$.

Lemma 2.5 *We posit assumptions (2.2), (2.4) and (2.6). Then F is an upper hemicontinuous with nonempty closed convex images*

If the **regulation map** $\Pi_M(\cdot)$ *is nontirivial, than the allocation set K is a viability domain of the set-valued map F defined by (2.8).*

Proof — First, since c is affine with respect to p and since the images $P(x)$ are convex and compact by assumption (2.4)i) and (2.6), we note that the images $F(x)$ are closed and compact.

Second, we remark that F is upper hemicontinuous. In order to check this point, we recall that

$$\sigma_{F(x)}(q) = \sup_{p\in P(x)} \langle q, c(x,p)\rangle$$

Since the map c is continuous by (2.4), the graph of P is closed by assumption (2.6) and the images $P(x)$ remain in the compact set S^l, we infer that $x \mapsto \sigma_{F(x)}(q)$ is upper semicontinuous.

Second, assumptions (2.2) imply that the tangent cone to the set of allocations is equal to

$$T_K(x) := \left\{ v := (v_1,\ldots,v_n) \in \prod_{i=1}^n T_{L_i}(x) \;\middle|\; \sum_{i=1}^n v_i \in T_M\left(\sum_{i=1}^n x_i\right)\right\}$$

Since $\Pi_M(x) \neq \emptyset$, there exists $p \in P(x)$ such that

$$\sum_{i=1}^n c_i(x_i,p) \in T_M(\sum_{i=1}^n x_i)$$

Therefore assumption (2.4) iv) implies that for such a $p \in P(x)$, $c(x,p)$ belongs to $T_K(x)$. □

In summary, we shall use Lemma 2.5 to prove that Theorem 2.3 follows from the following general Equilibrium Theorem 3.1.

Naturally, these remarks apply when single-valued change functions are replaced by **change maps** $C_i : L_i \times S^l \rightsquigarrow Y$ and the dynamics of the evolution of the consumption is described by the set-valued controlled system

$$\begin{cases} i) & x_i'(t) \in C_i(x_i(t), p(t)) \quad (i = 1,\ldots,n) \\ ii) & p(t) \in P(x(t)) \end{cases}$$

Equilibria of this dynamical system are solutions $(\bar{x}, \bar{p})$ to the system of inclusions

$$\begin{cases} i) & 0 \in C_i(\bar{x}_i, \bar{p}) \quad (i = 1, \ldots, n) \\ ii) & \bar{p} \in P(\bar{x}) \end{cases} \tag{2.9}$$

We then associate the **regulation map** Π_M defined by

$$\forall\, x \in K,\ \Pi_M(x) := \left\{ p \in P(x) \mid 0 \in T_M\Big(\sum_{i=1}^n x_i\Big) - \sum_{i=1}^n C_i(x_i, p) \right\}$$

Recall that a set-valued map F is said to be **convex** if its graph is convex[10].

We shall assume that

$$\begin{cases} i) & C_i(x, p) \text{ is a closed convex set-valued map with respect to } p \\ ii) & C_i \text{ is upper hemicontinuous with convex compact images} \\ iii) & \forall\, x_i \in L_i,\ p \in \mathrm{Im}(P),\ C_i(x_i, p) \subset T_{L_i}(x_i) \end{cases} \tag{2.10}$$

Theorem 2.6 *We posit assumptions (2.2), (2.6) and (2.10). Then if the* **regulation map** $\Pi_M(\cdot)$ *is nontirivial, then there exists a viable equilibrium* $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p}) \in K \times S^l$.

As in the single-valued case, we introduce set-valued map C defined by

$$C(x, p) := \prod_{i=1}^n C_i(x_i, p)$$

and the set-valued map F defined by

$$F(x) := \bigcup_{p \in P(x)} C(x, p) \tag{2.11}$$

Lemma 2.7 *We posit assumptions (2.2), (2.6) and (2.10). Then F is an upper hemicontinuous with nonempty closed convex images*

If the **regulation map** $\Pi_M(\cdot)$ *is nontirivial, then the allocation set K is a viability domain of the set-valued map F defined by (2.11).*

[10] See Chapter 2 of [3, Aubin & Frankowska] for an exhaustive presentation of closed convex maps and closed convex processes, which are the "set-valued analogues" of continuous linear operators.

Proof — First, since the set-valued map C defined by

$$C(x,p) := \prod_{i=1}^{n} C_i(x_i,p)$$

is also a closed convex set-valued map with respect to p and since the images $P(x)$ are convex and compact by assumption (2.10)i),ii) and (2.6), we infer that the images

$$F(x) := \bigcup_{p \in P(x)} C(x,p)$$

are closed and compact. Indeed, if $v_i \in C(x,p_i)$ $(i = 1,2)$, then

$$\alpha_1 v_1 + \alpha_2 v_2 \in \alpha_1 C(x,p_1) + \alpha_2 C(x,p_2) \subset C(x, \alpha_1 p_1 + \alpha_2 p_2) \subset C(x, P(x))$$

Since

$$\|C(x,p)\| \leq \lambda_x \|p\|$$

we deduce that the values of F are closed: Indeed, if $v_n \in C(x,p_n)$ converges to some v and since $p_n \in P(x)$ remains in a compact set, a subsequence (again denoted by) p_n converges to some $p \in P(x)$. Hence $(x, p_n, v_n) \in$ Graph(P) converges to (x,p,v), and, since the graph of C is closed, we infer that $v \in C(x, P(x)) =: F(x)$.

Finally, we derive that F is upper hemicontinuous because

$$\sigma_{F(x)}(q) = \sup_{p \in P(x)} \sigma_{C(x,p)}(q)$$

Since the map C is upper hemicontinuous by (2.10), the graph of P is closed by assumption (2.6) and the images $P(x)$ remain in the compact set S^l, we infer that $x \mapsto \sigma_{F(x)}(q)$ is upper semicontinuous.

Hence F is an upper hemicontinuous with nonempty closed convex images.

One prove exactly as before that K is a viability domain of F.

Also, we can check also that in this case, the regulation map $\Pi_M(\cdot)$ is not trivial if and only if the "budgetary condition"

$$\sup_{q \in N_M(\sum_{i=1}^{n} x_i)} \inf_{p \in P(x)} \sigma^{\flat}_{C_i(x_i,p)}(q) \leq 0 \quad \square$$

In this case also, Lemma 2.7 allows to derive Theorem 2.6 from the following general Equilibrium Theorem 3.1.

3 The Equilibrium Theorem

3.1 Case of Convex Domains

Theorem 3.1 (Equilibrium Theorem) *Assume that X is a finite dimensional vector-space*[11] *and that $F : X \rightsquigarrow X$ is an upper hemicontinuous set-valued map with closed convex images.*

If $K \subset X$ is a convex compact viability domain of F, then it contains an equilibrium of F.

Proof — We proceed by contradiction, assuming that the conclusion is false. Hence, for any $x \in K$, 0 does not belong to $F(x)$. Since the images of F are closed and convex, the Hahn-Banach Separation Theorem implies that there exists $p_x \in X^\star$ such that $\sigma(F(x), p_x) < 0$.

By setting

$$\mathcal{V}_p \; := \; \{x \; \in K \mid \; \sigma(F(x), p) \; < \; 0\}$$

the negation of the existence of an equilibrium of F in K implies that K can be covered by the subsets $\mathcal{V}_p$ when p ranges over the dual of X. These subsets are open by the very definition of upper hemicontinuity of F. So K can be covered by n such open subsets $\mathcal{V}_{p_i}$. Let us consider a continuous partition of unity $(\alpha_i)_{i=1,\ldots,n}$ associated with this finite open covering and introduce the function $\varphi : K \times K \to \mathbf{R}$ defined by

$$\varphi(x, y) \; := \; \sum_{i=1}^{n} \alpha_i(x) < p_i, x - y >$$

Being continuous with respect to x and affine with respect to y, the assumptions of Ky Fan's Inequality (Theorem 1.3) are satisfied. Hence there exists $\overline{x} \in K$ such that for $\overline{p} := \sum_{i=1}^{n} \alpha_i(\overline{x}) p_i$ we have

$$\forall\, y \in K, \quad \varphi(\overline{x}, y) \; = < \overline{p}, \overline{x} - y > \leq \; 0$$

The above inequality means that $-\overline{p}$ belongs to the polar cone $T_K(\overline{x})^-$ of the convex subset K at $\overline{x}$.

Since K is a viability domain of F, there exists $v \in F(\overline{x}) \cap T_K(\overline{x})$, and thus

$$\sigma(F(\overline{x}), \overline{p}) \; \geq < \overline{p}, v > \geq \; 0$$

We set

$$I(\overline{x}) \; := \; \{i = 1, \ldots, n \mid \alpha_i(\overline{x}) \; > \; 0\}$$

[11] Actually, the proof we give shows that this fundamental theorem remains true for any Hausdorff locally convex topological vector space and in particular for a Banach space endowed with the weak topology.

which is not empty. Hence

$$\sigma(F(\overline{x}),\overline{p}) \leq \sum_{i\in I(\overline{x})} \alpha_i(\overline{x})\sigma(F(\overline{x}),p_i) \ < 0$$

because, for any $i \in I(\overline{x})$, $\alpha_i(\overline{x}) > 0$, and thus, $\overline{x}$ belongs to the subset $\mathcal{V}_{p_i}$, which means precisely that $\sigma(F(\overline{x}),p_i) < 0$. The latter inequality is then a contradiction of the previous one. □

By modifying slightly the proof of the Equilibrium Theorem, we can prove the existence of zeros of a set-valued map from a Banach space X to another Banach space Y.

Theorem 3.2 *Let K be a convex compact subset of a Banach space X and F be an upper hemicontinuous set-valued map with closed convex values from K to another Banach space Y.*

Let us consider also a continuous map $B : K \mapsto \mathcal{L}(X,Y)$. If K, F and B are related by the condition

$$\forall\, x \in K,\ \ F(x) \cap \overline{B(x)T_K(x)} \ \neq \ \emptyset$$

then

$$\begin{cases} i) & \exists\, \overline{x} \in K \quad \text{such that } 0 \in F(\overline{x}) \\ ii) & \forall\, y \in K,\ \exists\, \widehat{x} \in K \quad \text{such that } B(\widehat{x})y \in B(\widehat{x})\widehat{x} + F(\widehat{x}) \end{cases}$$

Proof— The proof of the existence of an equilibrium $\overline{x} \in K$ of F is the same as the one of the Equilibrium Theorem, where we define the function φ by

$$\varphi(x,y) \ := \ \sum_{i=1}^{n} \alpha_i(x) < p_i, B(x)(x-y) >$$

Ky Fan's Inequality thus implies the existence of $\overline{x} \in K$ such that

$$-B(\overline{x})^{\star}\overline{p} \ \in \ T_K(\overline{x})^{-}$$

so that, taking a sequence $u_n \in T_K(\overline{x})$ such that $B(\overline{x})u_n$ converges to some

$$v \ \in \ F(\overline{x}) \cap \overline{B(\overline{x})T_K(\overline{x})}$$

(which exists by the tangential condition), we infer that

$$\begin{cases} \sigma(F(\overline{x}),\overline{p}) \\ \geq < \overline{p}, v > = \ \lim_{n\to\infty} < \overline{p}, B(\overline{x})u_n > = \ \lim_{n\to\infty} < B(\overline{x})^{\star}\overline{p}, u_n > \geq \ 0 \end{cases}$$

This inequality is contradicted as in the proof of Theorem 3.1.

Take now $y \in K$ and introduce the set-valued map $G : X \rightsquigarrow Y$ defined by

$$G(x) := F(x) + B(x)(x - y)$$

which also satisfies the assumptions of our theorem. Then there exists a zero $\widehat{x} \in K$ of G, which is a solution to the inclusion $B(\widehat{x})y \in B(\widehat{x})\widehat{x} + F(\widehat{x})$. □

As an example, we derive the existence of a solution to the equation $f(\bar{x}) = 0$ where the solution $\bar{x}$ must belong to a compact convex subset K:

Theorem 3.3 *Let X and Y be Banach spaces, $K \subset X$ be a compact convex subset, $\Omega \supset K$ be an open neighborhood of K and $f : \Omega \mapsto Y$ be a continuously differentiable single-valued map. Assume that*

$$\forall\, x \in K, \ -f(x) \in f'(x)T_K(x)$$

Then there exists a solution $\bar{x} \in K$ to the equation $f(\bar{x}) = 0$. In particular, when $x_0 \in K$ is given, there exists a sequence of elements $x_n \in K$ satisfying

$$\forall\, n \geq 0, \ f'(x_n)(x_n - x_{n-1}) = -f(x_n)$$

i.e., the implicit version of the Newton algorithm.

Proof— We take $F(x) := \{f(x)\}$ and $B(x) = -f'(x)$ in Theorem 3.2. □

3.2 Case of Convex Images

We shall derive from the Viability Theorem a neighboring statement, where the assumption of the convexity of K is replaced by the convexity of the image $F(K)$ of K by F:

Theorem 3.4 (Equilibrium Theorem) *Assume that X is a finite dimensional vector-space and that $F : X \rightsquigarrow X$ is an upper hemicontinuous set-valued map with closed convex images.*

If $K \subset X$ is a compact subset such that $F(K)$ is convex and if there exists at least a viable solution to the differential inclusion $x' \in F(x)$, then there exists a viable equilibrium of F in K.

Naturally, the Viability Theorem below implies the existence of such a viable solution whenever K is a viability domain of F.

Proof — Assume that there is no equilibrium. Hence, this means that 0 does not belong to the closed convex subset $F(K)$, so that the Separation Theorem implies the existence of some $p \in X^\star$ and $\varepsilon > 0$ such that

$$\sup_{x\in K, v\in F(x)} < v, -p > = \sigma(F(K), -p) < -\varepsilon$$

Hence, let us take any viable solution $x(\cdot)$ to differential inclusion

$$\text{for almost all } t \geq 0, \ \ x'(t) \in F(x(t))$$

which exists by assumption. We deduce that

$$\forall\, t \geq 0, \ \ < -p, x'(t) > \leq -\varepsilon$$

so that, integrating from 0 to t, we infer that

$$\varepsilon\, t \ \leq < p, x(t) - x(0) >$$

But K being bounded, we thus derive a contradiction. □

We can even relax the assumption of the convexity of $F(K)$, as the following "ergodic" Theorem shows:

Theorem 3.5 *Let us assume that F is upper hemicontinuous with closed convex images and that $K \subset \mathrm{Dom}(F)$ is compact. If there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K such that*

$$\inf_{t>0} \frac{1}{t} \int_0^t \|x'(\tau)\| d\tau \ = \ 0$$

then there exists a viable equilibrium $\bar{x}$, i.e., a state $\bar{x} \in K$ solution to the inclusion $0 \in F(\bar{x})$.

Observe that it is sufficient that the lim inf of the Cesaro means $\frac{1}{t_n} \int_0^{t_n} \|x'(\tau)\| d\tau$ of the velocities of a viable solution is equal to 0 to derive the existence of an equilibrium.

Proof — Let us assume that there is no viable equilibrium, i.e., that for any $x \in K$, 0 does not belong to $F(x)$. Since the images of F are closed and convex, the Separation Theorem implies that there exists $p \in \Sigma$, the unit sphere, and $\varepsilon_p > 0$ such that $\sigma(F(x), -p) < -\varepsilon_p$. In other words, we can cover the compact subset K by the subsets

$$\mathcal{V}_p \ := \ \{ \, x \in K \mid \sigma(F(x), -p) < -\varepsilon_p \, \}$$

when p ranges over Σ. They are open thanks to the upper hemicontinuity of F, so that the compact subset K can be covered by q open subsets $\mathcal{V}_{p_j}$. Set $\varepsilon := \min_{i=1,\ldots,q} \varepsilon_{p_i} > 0$.

Consider now a viable solution to the differential inclusion $x' \in F(x)$, which exists by assumption. Hence, for any $t \geq 0$, $x(t)$ belongs to some $\mathcal{V}_{p_j}$, so that

$$-\|x'(t)\| \leq \langle -p_j, x'(t)\rangle \leq \sigma(F(x(t)), -p_j) < -\varepsilon$$

and thus, by integrating from 0 to t, we have proved that there exists $\varepsilon > 0$ such that, for all $t > 0$,

$$\varepsilon < \frac{1}{t}\int_0^t \|x'(\tau)\| d\tau$$

a contradiction of the assumption of the theorem. □

3.3 Fixed Point Theorems

We begin by showing that Theorem 3.1 implies the Kakutani Fixed Point Theorem[12], which is the set-valued version of the Brouwer Fixed Point Theorem. Therefore, these theorems are all equivalent to the Brouwer Fixed Point Theorem and the Ky Fan Inequality.

Theorem 3.6 (Kakutani Fixed Point Theorem) *Let K be a convex compact subset of a Banach space X and $G : X \leadsto K$ be an upper hemicontinuous set-valued map with nonempty closed convex values. Then G has a fixed point*[13] $\overline{x} \in K \cap G(\overline{x})$.

Proof — We set $F(x) := G(x) - x$, which is also upper hemicontinuous with convex values. Since K is convex, then $K - x \subset T_K(x)$, and since $G(K) \subset K$, we deduce that K is a viability domain of F because

$$F(x) \subset K - x \subset T_K(x)$$

[12] called Ky Fan's Fixed Point Theorem in infinite dimensional spaces.

The story began in 1910 with the Brouwer Fixed Point Theorem, which was proved later in 1926 via the Three Polish Lemma, the three Poles being Knaster, Kuratowski and Mazurkiewicz. Knaster saw the connection between Sperner's Lemma and the fixed point theorem, Mazurkiewicz provided a proof corrected by Kuratowski. The extension to Banach spaces was proved in 1930 by their colleague Schauder.

Von Neumann did need the set-valued version of this Fixed Point Theorem in game theory, which was proved by Kakutani in 1941.

[13] which can be regarded as an equilibrium for the discrete set-valued dynamical system $x_{n+1} \in G(x_n)$.

Hence there exists a viable equilibrium $\bar{x} \in K$ of F, which is a fixed point of G. □

Actually, we do not need to assume that G maps K to itself. It is enough to assume that K is a viability domain of $F := G - \mathbf{1}$, which can be written in the following form

$$\forall x \in K, \quad G(x) \cap (x + T_K(x)) \neq \emptyset \tag{3.1}$$

This leads to the following

Definition 3.7 (Inward & Outward Maps) *A map $G : K \rightsquigarrow X$ satisfying property (3.1) is said to be* inward. *It is called* outward *if*

$$\forall x \in K, \quad G(x) \cap (x - T_K(x)) \neq \emptyset$$

Since K is a viability domain of $F := G - \mathbf{1}$ when G is inward and of $F_{-} := \mathbf{1} - G$ when G is outward, and since the equilibria of F and F_{-} are fixed points of G, we obtain the useful

Theorem 3.8 *Let K be a convex compact subset of a Banach space X and $G : K \rightsquigarrow X$ be an upper hemicontinuous map with nonempty closed convex values. If G is either inward or outward, it has a fixed point*

$$\bar{x} \in K \cap G(\bar{x})$$

See Part I for a list of references.

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