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Dynamical Qualitative Simulation

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FOREWORD

The purpose of this paper is to revisit the QSIM algorithm introduced by Kuipers in **qualitative physics** for studying the qualitative evolution of solutions to a differential equation using techniques of set-valued analysis and viability theory. It describes Dordan's software. It operates on a class of differential equations called "replicator systems", which play an important role in biochemistry and biology. This software provides the monotonic cells and draws them on the screen of the computer for three-dimensional systems (the state subset being the probability simplex). It also supplies symbolically the transitions from one monotonic cell to the other ones. It also provides a LaTeXreport providing the list of qualitative cells, singling out qualitative equilibria and describing the discrete dynamical system.

Dynamical Qualitative Simulation

Jean-Pierre Aubin

Introduction

The purpose of this paper is to revisit the QSIM algorithm introduced by Kuipers for studying the qualitative evolution of solutions to a differential equation $x' = f(x)$ where the state x ranges over a closed subset K of a finite dimensional vector-space $X := \mathbf{R}^n$.

The **qualitative state** of a solution to the differential equation at a given time t is the knowledge of the monotonicity property of each component $x_i(t)$ of a solution $x(\cdot)$ to this differential equation, i.e., the knowledge of the sign of the derivatives $x_i'(t)$. Hence the **qualitative behavior** is the evolution of the qualitative states of the solution, i.e., the evolution of the vector of signs of the components of $x'(t) = f(x(t))$, which must be determined without solving the differential equation.

In order to denote the qualitative states and track down their evolution, we introduce the n-dimensional **confluence space** $\mathcal{R}^n$ defined by

$$\mathcal{R}^n \;:=\; \{\,-,\,0,\,+\,\}^n$$

the convex cones where

$$\mathbf{R}^n_a \;:=\; \{\, v \in \mathbf{R}^n \mid \text{sign of } (v_i) = a_i \,\}$$

and their closures

$$a\mathbf{R}^n_+ \;:=\; \{\, v \in \mathbf{R}^n \mid \text{sign of } (v_i) = a_i \text{ or } 0 \,\}$$

We shall study the qualitative behavior of the differential equation, i.e., the evolution of the functions $t \mapsto s_n(x'(t))$ associated to solutions $x(\cdot)$ of the differential equation. Furthermore, we shall track down the "**landmarks**", i.e., the states at which the monotonic behavior of the solutions is modified. But, instead of finding them *a posteriori* by following the qualitative behavior of a given solution, we shall find them *a priori*, without solving the dynamical system, neither qualitatively nor analytically.

In other words, the problem arises whether we can **map the differential equation $x' = f(x)$ to a discrete dynamical system $\Phi : \mathcal{R}^n \rightsquigarrow \mathcal{R}^n$ on the qualitative space $\mathcal{R}^n$.**

This is not always possible, and we have thus to define the class of differential equations which enjoy this property.

For studying the qualitative behavior of the differential equation, we introduce the "**monotonic cells**" defined by

$$K_a \;:=\; \{x \in K \mid f(x) \in \mathbf{R}^n_a\}$$

Indeed, the quantitative states $x(\cdot)$ evolving in a given monotonic cell K_a share the same monotonicity properties because, as long as $x(t)$ remains in K_a,

$$\forall\, i = 1, \ldots, n, \;\text{ sign of } \frac{dx_i(t)}{dt} \;=\; a_i$$

These monotonic cells are examples of what one can call "**qualitative cells**" of the subset K. In full generality, qualitative cells are subsets $K_a \subset K$ of a family of subsets covering K. The problem is then to check whether a family of qualitative cells is consistent with a differential equation $x' = f(x)$ in the sense that one can find a discrete dynamical system Φ mapping each cells to other ones such that every solution starting from one cell K_a arrives in one of the qualitative cells of the image $\Phi(K_a)$.

This is not always possible and we shall conclude this paper by an extension of a result of D. Saari on "chaos". Chaos here means the following property: Given any arbitrary infinite sequence of qualitative cells, there is always one solution which visits these cells in the prescribed order.

To the extent that qualitative cells describe phenomena in the framework of the model described by such a differential equation, this discrete dynamical system Φ provides **causality** relations, by specifying what are the phenomena caused by a given one. In this sense, we are able to deduce from the model "physical laws". This one of the main motivations which give the names to this topic: **qualitative physics.**

But before, we shall characterize the qualitative equilibria, which are the qualitative cells such that the solutions which arrive in this qualitative cell remain in this cell. We shall also single out the qualitative repellers, which are qualitative cells such that any solution which arrives in this qualitative cell must leave this cell in finite time. We shall then provide conditions insuring that the qualitative cells are not empty.

The theoretical results concerning the version of the QSIM algorithm are illustrated by a software due to Olivier Dordan. It operates on a class of differential equations called "replicator systems", which play an important role in biochemistry and biology. This software provides the monotonic cells and draws them on the screen of the computer for three-dimensional systems (the subset K being the probability simplex). It also supplies symbolically the transitions from one monotonic cell to the other ones. It finally provides a LaTeXreport providing the list of qualitative cells, singling out qualitative equilibria and describing the discrete dynamical system Φ.

1 Monotonic Cells

We posit the assumptions of the Viability Theorem for differential equations (called the Nagumo Theorem):

$$\begin{cases} i) & f \text{ is continuous with linear growth} \\ ii) & K \text{ is a closed viability domain} \end{cases} \tag{1.1}$$

Therefore, from every initial state $x_0 \in K$ stars a solution to the differential equation

$$x'(t) \;=\; f(x(t)) \tag{1.2}$$

viable (remaining) in K.

1.1 Monotonic Behavior of the Components of the State

For studying the qualitative behavior of the differential equation (1.2), i.e., the evolution of the functions $t \mapsto s_n(x'(t))$ associated with solutions $x(\cdot)$ of the differential equation, we split the viability domain K of the differential equation into 3^n "**monotonic cells**" K_a and "**large monotonic cells**" $\overline{K}_a$, defined by

$$K_a \ := \ \{x \in K \mid f(x) \in \mathbf{R}^n_a\} \ \& \ \overline{K}_a \ := \ \{x \in K \mid f(x) \in a\mathbf{R}^n_+\}$$

Indeed, the quantitative states $x(\cdot)$ evolving in a given monotonic cell K_a share the same monotonicity properties because, as long as $x(t)$ remains in K_a,

$$\forall \, i = 1, \ldots, n, \ \text{sign of } \frac{dx_i(t)}{dt} \ = \ a_i$$

The monotonic cell K_0 is then the set of equilibria[1] of the system, because $K_0 = \{ \, x \in K \mid f(x) = 0 \, \}$.

These monotonic cells are examples of **qualitative cells**, and, for this reason, often called qualitative cells.

Studying the qualitative evolution of the differential equation amounts to know the laws (if any) which govern the transition from one monotonic cell K_a to other cells **without solving the differential equation**.

In Kuipers terminology, the boundaries of the monotonic cells are called the "**landmarks**". They are naturally unkown and are derived through the formulas defining these monotonic cells. The forthcoming algorithms compute them before studying the transition properties from one cell to another one (or other ones)

These laws thus **reveal causality relations** between qualitative phenomena concealed in the dynamical system, by specifying the successors of each monotonic cells, and present a major interest in physics for making some sense out of the maze of qualitative properties.

First, we mention the following result due to O. Dordan, stating that starting from any monotonic cell, a solution either converges to an equilibrium or leaves the monotonic cell in finite time:

Theorem 1.1 *Assume that a monotonic cell K_a is not empty and bounded. Then, for any initial state $x_0 \in K_a$, either the solution leaves K_a in finite time or it converges to an equilibrium.*

[1] Such an equilibrium does exist whenever the viability domain K is convex and compact, thanks to the Brouwer-Ky Fan Theorem.

Proof — Assume that a solution $x(\cdot)$ remains in K_a for all nonnegative $t \geq 0$.

Let any i such that $a_i \neq 0$. Since

$$x_i(t) - x_i(0) \; = \; \int_0^t x_i'(\tau) d\tau$$

we deduce that $x_i(t)$ is monotone and bounded. Therefore, it converges to some number x_i when $t \to +\infty$.

Consequently, each component of the solution $x(\cdot)$ is either equal to 0 or converges, so that $x(t)$ converges to a limit, which is then an equilibrium of the dynamical system. □

1.2 Monotonic Behavior of Observations of the State

But before proceeding further, we shall generalize our problem — free of any mathematical cost — to take care of physical considerations.

Instead of studying the monotonicity properties of each component $x_i(\cdot)$ of the state of the system under investigation, which can be too numerous, we shall only study the monotonicity properties of m functionals $V_j(x(\cdot))$ on the state (for instance, energy or entropy functionals in physics, observations in control theory, various economic indexes in economics) which do matter.

The previous case is the particular case when we take the n functionals V_i defined by $V_i(x) := x_i$.

We shall assume for simplicity that these functionals V_j are continuously differentiable around the viability domain K.

We denote by $\mathbf{V}$ the map from X to $Y := \mathbf{R}^m$ defined by

$$\mathbf{V}(x) \; := \; (V_1(x), \ldots, V_m(x))$$

Since the derivative of the observation $\mathbf{V}(x(\cdot))$ is equal to $\mathbf{V}'(x(\cdot))x'(\cdot) = \mathbf{V}'(x(\cdot))f(x(\cdot))$, it will be convenient to set

$$\forall \, x \in K, \;\; g(x) \; := \; \mathbf{V}'(x)f(x)$$

Hence, we associate with each qualitative state a the qualitative cells K_a and the large qualitative cells $\overline{K}_a$ defined by

$$K_a \; := \; \{x \in K \mid g(x) \in \mathbf{R}^m_a\} \;\; \& \;\; \overline{K}_a \; := \; \{x \in K \mid g(x) \in a\mathbf{R}^m_+\}$$

In other words, the quantitative states $x(\cdot)$ evolving in a given monotonic cell K_a share the same monotonicity properties of their observations because, as long as $x(t)$ remains in K_a,

$$\forall\, j = 1, \ldots, m, \ \text{sign of } \frac{d}{dt}V_j(x(t)) \ = \ a_j$$

In particular, the m functions $V_j(x(t))$ remain constant while they evolve in the qualitative cell K_0.

By using observation functionals chosen in such a way that many qualitative cells are empty, the study of transitions may be drastically simplified: this is a second reason to carry our study in this more general setting.

This is the case for instance when the observation functionals are "Lyapunov functions" $V_j : K \mapsto \mathbf{R}$. We recall that V is a Lyapunov function if $< V'(x), f(x) > \leq 0$ for all $x \in K$, so that $V(x(\cdot))$ decreases along the solutions to the differential equation.

Hence, if the observation functionals are Lyapunov functions, the qualitative cells K_a are empty whenever a component a_i is positive. In this case, we have at most 2^m nonempty qualitative cells. (In some sense, one can say that Lyapunov was the originator of qualitative simulation a century ago).

Naturally, we would like to know directly the laws which govern the transition from one qualitative cell K_a to other qualitative cells, without solving the differential equation, and therefore, without knowing the state of the system, but only some of its properties.

2 Transitions Between Qualitative Cells

We shall assume from now on that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K.

Let us denote by $S : K \mapsto \mathcal{C}^1(0, \infty; X)$ the "solution map" associating with each initial state $x_0 \in K$ the solution $Sx_0(\cdot)$ to the differential equation (1.2) starting at x_0.

Definition 2.1 *Let us consider a map f from K to X and m observation functionals $V_j : K \mapsto \mathbf{R}$. We denote by $\mathcal{D}(f, \mathbf{V})$, the subset of* *qualitative states $a \in \mathcal{R}^n$ such that the associated qualitative cell K_a is not empty.*

We shall say that a qualitative state $c \in \mathcal{D}(f, \mathbf{V})$ is a "successor" of $b \in \mathcal{D}(f, \mathbf{V})$ if for all initial states $x_0 \in \overline{K}_b \cap \overline{K}_c$, there exists $\tau \in]0, +\infty]$ such that $Sx_0(s) \in K_c$ for all $s \in]0, \tau[$.

A qualitative state $a \in \mathcal{D}(f, \mathbf{V})$ is said to be a "qualitative equilibrium" if it is its own successor. It is said to be a "qualitative repellor" if for any initial state $x_0 \in \overline{K}_a$, there exists $t > 0$ such that $Sx_0(t) \notin \overline{K}_a$.

Our first objective is to express the fact that c is a successor of b through a set-valued map Φ.

For that purpose, we shall set

$$h(x) := g'(x)f(x) = V''(x)(f(x), f(x)) + V'(x)f'(x)f(x)$$

We introduce the notation

$$\overline{K}_a^i := \{ x \in \overline{K}_a \mid g(x)_i = 0 \}$$

(Naturally, $\overline{K}_a = \overline{K}_a^i$ whenever $a_i = 0$.)

We shall denote by Γ the set-valued map from $\mathcal{R}^m$ to itself defined by

$$\forall\, a \in \mathcal{R}^m,\ (\Gamma(a))_i \text{ is the set of signs of } h_i(x) \text{ when } x \in \overline{K}_a^i$$

We also set $I_0(x) := \{ i = 1, \dots m \mid g(x)_i = 0 \}$ and

$$\mathbf{R}_+^{I_0(x)} := \{ v \in \mathbf{R}^m \mid v_i \geq 0 \ \ \forall\, i \in I_0(x) \}$$

We introduce the operations $\wedge$ on $\mathcal{R}^m$ defined by

$$(b \wedge c)_i := \begin{cases} b_i & \text{if } b_i = c_i \\ 0 & \text{if } b_i \neq c_i \end{cases}$$

and the set-valued operation $\vee$ where $b \vee c$ is the subset of qualitative states a such that

$$a_i := b_i \text{ or } c_i$$

We set

$$a \# b \iff \forall\, i = 1, \dots, m,\ a_i \neq b_i$$

Proposition 2.2 *The set-valued map* Γ *satisfies the consistency property*

$$\Gamma(a \vee 0) \quad \subset \quad \Gamma(a)$$

and thus,

$$\Gamma(b \wedge c) \quad \subset \quad \Gamma(b) \cap \Gamma(c)$$

Proof — To say that $\overline{K}_b$ is contained in $\overline{K}_a$ amounts to saying that b belongs to $a \vee 0$. When this is the case, we deduce that for all $i = 1, \ldots, m$, $\overline{K}_b^i \subset \overline{K}_a^i$, so that the signs taken by $h(x)_i$ when x ranges over $\overline{K}_b^i$ belong to the set of $\Gamma(a)_i$ of signs taken by the same function over $\overline{K}_a^i$. Therefore, $\Gamma(b)$ is contained in $\Gamma(a)$.

Since $b \wedge c$ belongs to both $b \vee 0$ and $c \vee 0$, we deduce that $\Gamma(b \wedge c)$ is contained in both $\Gamma(b)$ and $\Gamma(c)$. □

Definition 2.3 *We shall associate with the system* $(f, \mathbf{V})$ *the discrete dynamical system on the confluence set* $\mathcal{R}^m$ *defined by the set-valued map* $\Phi : \mathcal{R}^m \rightsquigarrow \mathcal{R}^m$ *associating with any qualitative state* b *the subset*

$$\Phi(b) \quad := \quad \{\, c \in \mathcal{D}(f, \mathbf{V}) \mid \Gamma(b \wedge c) \quad \subset \quad c \vee 0 \,\}$$

We begin with necessary conditions for a qualitative state c to be a successor of b:

Proposition 2.4 *Let us assume that* f *is continuously differentiable and that the* m *functions* V_j *are twice continuously differentiable around the viability domain* K.

If $c \in \mathcal{D}(f, \mathbf{V})$ *is a successor of* b, *then* c *belongs to* $\Phi(b)$.

Before proving this proposition, we need the following

Lemma 2.5 *Let us assume that* f *is continuously differentiable and that the* m *functions* V_j *are twice continuously differentiable around the viability domain* K.

If v *belongs to the contingent cone to the* $\overline{K}_a$ *at* x, *then condition*

$$v \in T_K(x) \;\;\&\;\; \forall\, i \in I_0(x), \;\; \text{sign of } \; (g'(x)v)_i = a_i \;\; \text{or} \;\; 0$$

is satisfied.

The converse is true if we posit the transversality assumption:

$$\forall\, x \in \overline{K}_a, \;\; g'(x)C_K(x) - a\mathbf{R}_+^{I_0(x)} \;\; = \;\; \mathbf{R}^m$$

Proof — Since the large qualitative cell $\overline{K}_a$ is the intersection of K with the inverse image by g of the convex cone $a\mathbf{R}^m_+$, we know that the contingent cone to $\overline{K}_a$ at some $x \in \overline{K}_a$ is contained in

$$T_K(x) \cap g'(x)^{-1} T_{a\mathbf{R}^m_+}(g(x))$$

and is equal to this intersection provided that the "transversality assumption"

$$g'(x)C_K(x) - C_{a\mathbf{R}^m_+}(g(x)) \;=\; \mathbf{R}^m$$

is satisfied. On the other hand, we know that $a\mathbf{R}^m_+$ being convex,

$$C_{a\mathbf{R}^m_+}(y) \;=\; T_{a\mathbf{R}^m_+}(y) \;=\; aT_{\mathbf{R}^m_+(ay)} \supset a\mathbf{R}^m_+$$

and that $v \in T_{\mathbf{R}^m_+}(z)$ if and only if

$$\text{whenever } z_j = 0, \text{ then } \; v_j \geq 0$$

Consequently, $v \in T_{a\mathbf{R}^m_+}(g(x))$ if and only if

$$\text{whenever } g(x)_j = 0, \text{ then sign of } \; v_j \;=\; a_j \; \text{ or } \; 0$$

i.e., $T_{a\mathbf{R}^m_+}(g(x)) = a\mathbf{R}^{I_0(x)}_+$.

Hence v belongs to the contingent cone to $\overline{K}_a$ at x if and only if v belongs to $T_K(x)$ and $g'(x)v$ belongs to $T_{a\mathbf{R}^m_+}(g(x))$, i.e., the sign of $(g'(x)v)_j$ is equal to a_j or 0 whenever j belongs to $I_0(x)$. □

Proof of Proposition 2.4 — Let c be a successor of b. Take any initial state x_0 in $\overline{K}_b \cap \overline{K}_c$ and set $x(t) := Sx_0(t)$. We observe that the intersection of two qualitative cells $\overline{K}_b$ and $\overline{K}_c$ is equal to

$$\overline{K}_b \cap \overline{K}_c \;:=\; \overline{K}_{b \wedge c}$$

Since the solution $x(t)$ to the differential equation crosses the intersection $\overline{K}_{b\wedge c}$ towards $\overline{K_c}$, $f(x_0)$ belongs to the contingent cone $T_{\overline{K}_c}(x_0)$ because

$$\liminf_{h \to 0+} d_{K_c}(x_0 + hf(x_0))/h \leq \liminf_{h \to 0+} \left\| x'(0) - \frac{x(h) - x_0}{h} \right\| \;=\; 0$$

By Lemma 2.5, this implies that

$$\forall\, x_0 \in \overline{K}_{b\wedge c},\ \ \forall\, i \in I_0(x_0),\ \ \text{sign of}\, h_i(x_0) \ = \ c_i \ \text{ or } \ 0$$

or, equivalently, that

$$\Gamma(b \wedge c) \ \subset \ c \vee 0$$

Hence c belongs to $\Phi(b)$, as it was announced. □

3 Qualitative Equilibrium and Repellor

We can characterize the qualitative equilibria of differential equation (1.2).

Theorem 3.1 *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. We posit the transversality assumption*

$$\forall\, x \in \overline{K}_a,\ \ g'(x)C_K(x) - a\mathbf{R}_+^{I_0(x)} \ = \ \mathbf{R}^m$$

Then a is a qualitative equilibrium if and only if a belongs to $\Phi(a)$.

Proof— We already know that if a is a qualitative equilibrium, then a belongs to $\Phi(a)$. We shall prove the converse statement, and, for that purpose, observe that saying that a is a qualitative equilibrium amounts to saying that the large qualitative cell $\overline{K}_a$ enjoys the viability property (or is invariant by f). By the Nagumo Theorem, this is equivalent to say that $\overline{K}_a$ is a viability domain, i.e., that

$$\forall\, x \in \overline{K}_a,\ \ f(x) \ \in \ T_{\overline{K}_a}(x)$$

By Lemma 2.5, knowing that $f(x)$ belongs to the contingent cone $T_K(x)$ by assumption, this amounts to say that

$$\forall\, x \in \overline{K}_a,\ \ \forall\, i \in I_0(x),\ \ \text{sign of}\ \ (g'(x)f(x))_i = a_i \ \text{ or } \ 0$$

i.e., that $\Gamma(a \wedge a) = \Gamma(a) \subset a \vee 0$. Hence, a is a fixed point of Φ. □

When a large qualitative cell $\overline{K}_a$ is not a viability domain of f, i.e., if a is not a qualitative equilibrium, at leat a solution leaves the qualitative cell in finite time and thus, will reach the boundary of this cell in finite time.

We infer from the definition of the viability kernel that

Proposition 3.2 *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. We posit the transversality assumption*

$$\forall\, x \in \overline{K}_a, \;\; g'(x)C_K(x) - a\mathbf{R}_+^{I_0(x)} \;=\; \mathbf{R}^m$$

- *The qualitative state a is a qualitative repellor if and only if the viability kernel of $\overline{K}_a$ is empty.*
- *If for some $b \in a \vee 0$, the qualitative cell $\overline{K}_b$ is contained in the viability kernel* $\mathrm{Viab}(\overline{K}_a)$*, then a is the only successor of b.*

Proof

1 — To say that some $x_0 \in \overline{K}_a$ does not belong to the viability kernel of $\overline{K}_a$ means that for some $t > 0$, $s(t)x_0 \notin \overline{K}_a$. If this happens for all $x_0 \in \overline{K}_a$, then obviously, a is a qualitative repellor.

2 — If $\overline{K}_b \subset \mathrm{Viab}(\overline{K}_a)$, then, for all $x_0 \in \overline{K}_b$, $s(t)x_0 \in \overline{K}_a$ for all $t \geq 0$. Hence a is the only successor of b. □

4 The QSIM Algorithm

We shall now distinguish the 2^n "full qualitative states" $a\#0$ from the other qualitative states, the "transition states".

When I is a non empty subset of $N := \{\, 1, \ldots, m \,\}$, we associate with a full state $a\#0$ the transition state a^I defined by

$$a_i^I \;:=\; \begin{cases} 0 & \text{if } i \in I \\ a_i & \text{if } i \notin I \end{cases}$$

What are the successors, if any, of a given transition state a^I?

This question does not always receive an answer, since, starting from some initial state $x \in K_a^I$, there may exist two sequences $t_n > 0$ and $s_n > 0$ converging to $0+$ such that $x(t_n) \in \overline{K}_a$ and $x(s_n) \notin \overline{K}_a$

We can exclude this pathological phenomenon in two instances.

One obviously happens when either a or the transition state a^I is an equilibrium, i.e., when

$$\Gamma(a)_i = 0 \text{ for } i \in I \text{ and } \Gamma(a)_i \subset \{a_i, 0\} \text{ for } i \notin I$$

This also happens in the following situation:

Lemma 4.1 *Let $a\#0$ be a full transition state. If $\Gamma(a)\#0$ (and thus, is reduced to a point) then, for any transition state a^I, there exists a unique successor $b := \Phi(a^I)\#0$, i.e., for all initial states x in the transition cell K_{a^I} there exists $t_2 > 0$ such that, for all $t \in]0, t_2[$, the solution $x(t)$ remains in the full qualitative cell K_b.*

Proof — We consider an initial state $x \in K_{a^I}$.

If $i \notin I$, then the sign of $g(x)_i$ is equal to $a_i \neq 0$, and thus, there exists $\eta_i > 0$ such that the sign of $g(x(t)_i)$ remains equal to $a_i^I = a_i$ when $t \in [0, \eta_i[$.

If $i \in I$, then $g(x)_i = 0$, and we know that the sign of the derivative $\frac{d}{dt} g_i(x(t))_{|t=0} = h_i(x)$ is equal to $\Gamma(a)_i$ and is different from 0. Hence there exists $\eta_i > 0$ such that the sign of $h(x(t))_i$ remains equal to b_i when $t \in]0, \eta_i[$, so that the sign of

$$g_i(x(t)) = \int_{t_1}^{t} h_i(x(\tau)) d\tau$$

remains equal to $\Gamma(a)_i$ on the interval $]0, \eta_i[$.

Hence we have proved that there exists some $\eta > 0$ such that $x(t) \in K_b$ for $t \in]0, t_2[$ where

$$b_i := \begin{cases} \Gamma(a)_i & \text{when } i \in I \\ a_i & \text{when } i \notin I \end{cases}$$

and where $t_2 := \min_i \eta_i > 0$. □

Definition 4.2 *We shall say that the system $(f, \mathbf{V})$ is "strictly filterable" if and only if for all full state $a \in \mathcal{D}(f, \mathbf{V})\#0$, either $\Gamma(a)\#0$ or a is a qualitative equilibrium or all the transition states a^I $(I \neq \emptyset)$ are qualitative equilibria.*

We deduce from Definition 4.2 and the above observations the following consequence:

Theorem 4.3 *Let us assume that f is continuously differentiable, that the m functions V_j are twice continuously differentiable around the viability domain K and that the system $(f, \mathbf{V})$ is "strictly filterable". Let $a \in \mathcal{R}^m$ be an initial full qualitative state.*

Then, for any initial state $x \in K_a$, the sign vector

$$a_x(t) \;:=\; s_m\left(\frac{d}{dt}(\mathbf{V}(Sx(t)))\right)$$

is a solution to the QSIM algorithm defined in the following way:

There exist a sequence of qualitative states a_k satisfying

$$a_0 := a \;\;\&\;\; a_{k+1} \;\in\; \Phi(a_k \vee 0) \tag{4.1}$$

and a sequence of landmarks $t_0 := 0 < t_1 < \ldots < t_n < \ldots$ such that

$$\begin{cases} \forall\, t \in]t_k, t_{k+1}[, \;\; a(t) \;=\; a_k \\ a(t_{k+1}) \;=\; a_k \wedge a_{k+1} \end{cases} \tag{4.2}$$

In other words, we know that the vector signs of the variations of the observations of the solutions to differential equation (1.2) evolve according the set-valued dynamical system (4.1) and stop when a_k is either a qualitative equilibrium or all its transition states a_k^I are qualitative equilibria.

Remark — The solutions to the QSIM algorithm (4.1) do not necessarily represent the evolution of the vector signs of the variations of the observations of a solution to the differential equation.

Further studies must bring answers allowing to delete impossible transitions from one full qualitative cell $\overline{K}_a$ to some of its transition cells K_{a^I}.

This is the case of a qualitative equilibrium, for instance, since a is the only successor of itself. □

Therefore, the QSIM algorithm requires the definition of the set-valued map $\Gamma : \mathcal{R}^m \leadsto \mathcal{R}^m$ by computing the signs of the m functions $h_i(\cdot)$ on the qualitative cells K_a^i for all $i \in N$ and $a \in \mathcal{D}(f, \mathbf{V})\#0$.

If by doing so, we observe that the system is strictly filterable, then we know that the set-valued dynamical system (4.1) contains the evolutions of the vector signs of the m observations of solutions to the differential equation (1.2).

5 Replicator Systems

We begin by studying the viability property of the probability simplex

$$S^n := \left\{ x \in \mathbf{R}^n_+ \mid \sum_{i=1}^{n} x_i = 1 \right\}$$

This is the most important example, because, in many problems, it is too difficult to describe mathematically the state of the system. Then, assuming there is a finite number n of states, one rather study the evolution of their frequencies, probabilities, concentrations, mixed strategies (in games), etc.... instead of the evolution of the state itself. We shall provide examples later in this section.

We refer to the first Chapter of VIABILITY THEORY for more details about the replicator systems, which are studied in depth in the book THE THEORY OF EVOLUTION AND DYNAMICAL SYSTEMS by J. Hofbauer and K. Sigmund.

The contingent cone $T_{S^n}(x)$ to S^n at $x \in S^n$ is the cone of elements $v \in \mathbf{R}^n$ satisfying

$$\sum_{i=1}^{n} v_i = 0 \quad \& \quad v_i \geq 0 \quad \text{whenever} \quad x_i = 0 \tag{5.1}$$

(See Appendix A-7)

We shall investigate now how to make viable the evolution of a system for which we know the growth rates $g_i(\cdot)$ of the evolution without constraints (also called "specific growth rates"):

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) = x_i(t) g_i(x(t))$$

There are no reasons[2] for the solutions to this system of differen-

[2]By Nagumo's Theorem, the functions g_i should be continuous and satisfy:

$$\forall\, x \in S^n, \quad \sum_{i=1}^{n} x_i g_i(x) \; = \; 0$$

tial equations to be viable in the probability simplex.

But we can correct it by substracting to each initial growth rate the common "feedback control $\tilde{u}(\cdot)$" (also called "global flux" in many applications) defined as the weighted mean of the specific growth rates

$$\forall\, x \in S^n, \;\; \tilde{u}(x) \;:=\; \sum_{j=1}^{n} x_j g_j(x)$$

Indeed, the probability simplex S^n is obviously a viability domain of the new dynamical system, called "replicator system" (or system "under constant organization"):

$$\begin{cases} \forall\, i = 1,\ldots,n, \; x_i'(t) & = x_i(t)(g_i(x(t)) - \tilde{u}(x(t))) \\ & = x_i(t)\left(g_i(x(t)) - \displaystyle\sum_{j=1}^{n} x_j(t) g_j(x(t))\right) \end{cases} \tag{5.2}$$

An equilibrium α of the replicator system (5.2) is a solution to the system

$$\forall\, i = 1,\ldots,n, \;\; \alpha_i(g_i(\alpha) - \tilde{u}(\alpha)) = 0$$

(Such an equilibrium does exist, thanks to the Equilibrium Theorem). These equations imply that either $\alpha_i = 0$ or $g_i(\alpha) = \tilde{u}(\alpha)$ or both, and that $g_{i_0}(\alpha) = \tilde{u}(\alpha)$ holds true for at least one i_0. We shall say that an **equilibrium α is non degenerate** if

$$\forall\, i = 1,\ldots,n, \quad g_i(\alpha) \;=\; \tilde{u}(\alpha) \tag{5.3}$$

Equilibria α which are strongly positive (this means that $\alpha_i > 0$ for all $i = 1,\ldots,n$) are naturally non degenerate.

We associate[3] with any $\alpha \in S^n$ the function V_α defined on the

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[3]The reason why we introduce this function is that α is the unique maximizer of V_α on the simplex S^n. This follows from the convexity of the function $\varphi := -\log$: Setting $0 \log 0 = 0 \log \infty = 0$, we get

$$\sum_{i=1}^{n} \alpha_i \log \frac{x_i}{\alpha_i} = \sum_{\alpha_i > 0} \alpha_i \log \frac{x_i}{\alpha_i} \leq \log\Big(\sum_{\alpha_i > 0} x_i\Big) \leq \log 1 = 0$$

simplex S^n by

$$V_\alpha(x) := \prod_{i=1}^{n} x_i^{\alpha_i} := \prod_{i\in I_\alpha} x_i^{\alpha_i}$$

where we set $0^0 := 1$ and $I_\alpha := \{i = 1, \ldots, n \mid \alpha_i > 0\}$.

Let us denote by S^I the subsimplex of elements $x \in S^n$ such that $x_i > 0$ if and only if $i \in I$.

Theorem 5.1 *Let us consider n continuous growth rates g_i. For any initial state $x_0 \in S^n$, there exists a solution to replicator system (5.2) starting from x_0 and which is viable in the subsimplex $S^{I_{x_0}}$.*

The viable solutions satisfy

$$\forall\, t \geq 0, \ \sum_{i=1}^{n} g_i(x(t))x_i'(t) \ \geq \ 0 \tag{5.4}$$

and, whenever $\alpha \in S^n$ is a nondegenerate equilibrium,

$$\frac{d}{dt}V_\alpha(x(t)) \ = \ -V_\alpha(x(t))\sum_{i=1}^{n}(x_i(t)-\alpha_i)(g_i(x(t))-g_i(\alpha)) \tag{5.5}$$

Proof — We first observe that

$$\forall\, x \in S^{I_{x_0}}, \ \sum_{i\in I_{x_0}} x_i(g_i(x)-\tilde{u}(x)) \ = \ 0$$

because, $x_i = 0$ whenever $i \notin I_{x_0}$, i.e., whenever $x_{0_i} = 0$. Therefore, the subsimplex $S^{I_{x_0}}$ is a viability domain of the replicator system (5.2).

Inequality (5.4) follows from Cauchy-Schwarz inequality because

$$\left(\sum_{i=1}^{n} x_i g_i(x)\right)^2 \leq \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_i g_i(x)\right)^2 = \sum_{i=1}^{n} x_i g_i(x)^2$$

so that

$$\sum_{i=1}^{n} \alpha_i \log x_i \ \leq \ \sum_{i=1}^{n} \alpha_i \log \alpha_i$$

and thus, $V_\alpha(x) \leq V_\alpha(\alpha)$ with equality if and only if $x = \alpha$.

We deduce formula (5.5) from

$$\begin{cases} \frac{d}{dt}V_\alpha(x(t)) \;=\; \sum_{i\in I_\alpha} \frac{\partial}{\partial x_i}V_\alpha(x(t))x_i'(t) \\ \\ =\; V_\alpha(x(t))\displaystyle\sum_{i\in I_\alpha}\alpha_i\frac{x_i'(t)}{x_i(t)} \;=\; V_\alpha\sum_{i=1}^{n}\alpha_i\frac{x_i'(t)}{x_i(t)} \end{cases}$$

and from

$$\sum_{i=1}^{n}\alpha_i\frac{x_i'(t)}{x_i(t)} \;=\; \sum_{i=1}^{n}(\alpha_i - x_i(t))g_i(x(t))$$

Then we take into account that α being a non degenerate equilibrium, because inequality (5.3) implies that

$$\sum_{i=1}^{n}(\alpha_i - x_i(t))g_i(\alpha) \;=\; 0 \quad \square$$

Remark — When the specific growth rates are derived from a differentiable potential function U by

$$\forall\, i = 1,\ldots,n,\;\; g_i(x) \;:=\; \frac{\partial U}{\partial x_i}(x)$$

condition (5.4) implies that

$$\forall\, t \geq 0,\;\; \frac{dU}{dt}(x(t)) \;\geq\; 0$$

because

$$\frac{dU}{dt}(x(t)) = \sum_{i=1}^{n}\frac{\partial U}{\partial x_i}(x(t))x_i'(t) = \sum_{i=1}^{n}g_i(x(t))x_i'(t) \geq 0$$

Therefore, **the potential function U does not decrease along the viable solutions to the replicator system** (5.2).

Furthermore, when this potential function U is homogeneous with degree p, Euler's formula implies that

$$\tilde{u}(x) \;=\; pU(x)$$

(because $\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}U(x) \;=\; pU(x)$) so that in this case, **the global flux $\tilde{u}(x(t))$ also does not decrease along the viable solutions to the replicator system** (5.2).

On the other hand, if we assume that the growth rates g_i are "decreasing" in the sense that

$$\forall\, x, y \in S^n, \quad \sum_{i=1}^{n}(x_i - y_i)(g_i(x) - g_i(y)) \;\leq\; 0$$

then inequality (5.5) implies that for any non degenerate equilibrium $\alpha \in S^n$,

$$\forall\, t \geq 0, \quad \frac{dV_\alpha}{dt}(x(t)) \;\geq\; 0$$

When $g(x) := U'(x)$ is derived from a concave differentiable potential U, it is decreasing so that, **for a concave potential, both** $U(x(\cdot))$ **and** $V_\alpha(x(\cdot))$ **are increasing.** □

Example: Replicator systems for constant growth rates. The simplest example is the one where the specific growth rates $g_i(\cdot) \equiv a_i$ are constant. Hence we correct constant growth systems $x_i' = a_i x_i$ whose solutions are exponential $x_{0_i}e^{a_i t}$, by the 0-order replicator system

$$\forall\, i = 1, \ldots, n, \quad x_i'(t) \;=\; x_i(t)(a_i - \sum_{j=1}^{n} a_j x_j(t))$$

whose solutions are given explicitly by:

$$x_i(t) \;=\; \frac{x_{0_i}e^{a_i t}}{\sum_{j=1}^{n} x_{0_j}e^{a_j t}} \quad \text{whenever} \quad x_{0_i} > 0$$

(and $x_i(t) \equiv 0$ whenever $x_{0_i} = 0$).

Example: Replicator systems for linear growth rates. The next class of examples is provided by linear growth rates

$$\forall\, i = 1, \ldots, n, \quad g_i(x) \;:=\; \sum_{j=1}^{n} a_{ij} x_j$$

Let A denote the matrix the entries of which are the above a_{ij}'s. Hence the global flux can be written

$$\forall\, x \in S^n, \quad \tilde{u}(x) \;=\; \sum_{k,l=1}^{n} a_{kl} x_k x_l \;=\; < Ax, x >$$

Hence, first order replicator systems can be written[4].

$$\forall\, i = 1,\ldots,n,\;\; x_i'(t) \;=\; x_i(t)(\sum_{j=1}^{n} a_{ij}x_j(t) - \sum_{k,l=1}^{n} a_{kl}x_k(t)x_l(t))$$

Such systems have been investigated independently in

— population genetics (allele frequencies in a gene pool)

— theory of prebiotic evolution of selfreplicating polymers (concentrations of polynucleotides in a dialysis reactor)

— sociobiological studies of evolutionary stable traits of animal behavior (distributions of behavioral phenotypes in a given species)

— population ecology (densities of interacting species)

6 Qualitative Simulation of Replicator Systems

Qualitative analysis of replicator systems had been carried out by Olivier Dordan, who designed a software which provides the transition matrix, qualitative equilibria and repellors of any first-order replicator system. In the three dimensional case, the computer program draws the qualitative cells in the two-dimensional simplex S^3.

Let A denote the matrix the entries of which are a_{ij}. First order replicator systems can be written

$$\forall\, i = 1,\ldots,n,\;\; x_i'(t) \;=\; x_i(t)\left(\sum_{j=1}^{n} a_{ij}x_j(t) - \sum_{k,l=1}^{n} a_{kl}x_k(t)x_l(t)\right) \tag{6.1}$$

We infer that **the boundaries of the qualitative cells are quadratic manifolds**, since they are given by the equations

$$\forall\, i = 1,\ldots,n,\;\; \sum_{j=1}^{n} a_{ij}x_j - \sum_{k,l=1}^{n} a_{kl}x_k(t)x_l \;=\; 0$$

When the matrix A is entered in the software, it computes the qualitative cells (and thus, supplies all the landmarks), singles out

[4]Observe that if for each i, all the a_{ij} are equal to b_i, we find 0-order replicator systems

the qualitative equilibria and furnishes symbolically the qualitative transition map Φ.

It also delivers LaTeXreports such these ones:

Example 1 Let the matrix A involved in the replicator system (6.1)

$$A = \begin{pmatrix} 1.00 & 2.00 & -1.00 \\ .00 & .00 & -2.00 \\ 2.00 & .00 & 1.00 \end{pmatrix}$$

Qualitative results

There are 2 nonempty
full" qualitative cells.

Computation of the qualitative system Φ

$$\begin{bmatrix} \Phi(-,-,+) & = & (-,-,+) \\ \Phi(+,-,+) & = & (-,-,+) \end{bmatrix}$$

The following qualitative set is a qualitative equilibrium

$$(-,-,+)$$

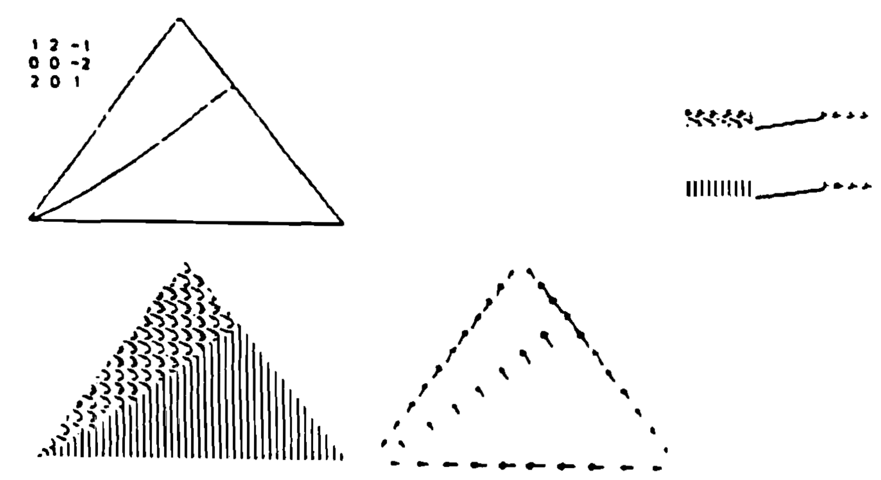
Computation of the set-valued map Γ

$$\begin{array}{lcl} \Gamma(-,-,+) & = & (\{-\},\{\emptyset\},\{\emptyset\}) \\ \Gamma(0,-,+) & = & (\{-\},\{\emptyset\},\{\emptyset\}) \\ \Gamma(0,0,0) & = & (\{0\},\{0\},\{\emptyset\}) \\ \Gamma(+,-,+) & = & (\{-\},\{\emptyset\},\{\emptyset\}) \end{array}$$

Example 2 Let the matrix A involved in the replicator system (6.1)

$$A = \begin{pmatrix} 1.00 & 2.00 & -1.00 \\ 3.00 & .00 & -2.00 \\ 2.00 & .00 & 1.00 \end{pmatrix}$$

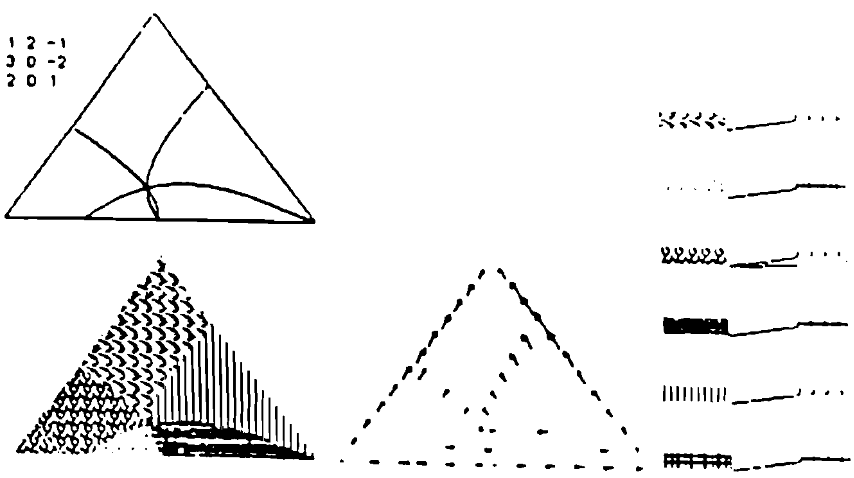
Figure 1: Qualitative Simulation of Replicator Systems # 1



Example of qualitative cells of a replicator system for the matrix

$$A = \begin{pmatrix} 1.00 & 2.00 & -1.00 \\ .00 & .00 & -2.00 \\ 2.00 & .00 & 1.00 \end{pmatrix}$$

Figure 2: Qualitative Simulation of Replicator Systems # 2



Example of qualitative cells of a replicator system for the matrix

$$A = \begin{pmatrix} 1.00 & 2.00 & -1.00 \\ 3.00 & .00 & -2.00 \\ 2.00 & .00 & 1.00 \end{pmatrix}$$

Qualitative results

There are 6 nonempty "full" qualitative cells.

Computation of the qualitative system Φ

$$\left[\begin{array}{lcl}
\Phi(-,-,+) & = & (-,-,+) \\
\Phi(+,-,+) & = & (-,-,+) \\
\Phi(-,+,-) & = & (+,+,-) \\
\Phi(-,+,+) & = & \left\{\begin{array}{l}(-,-,+)\\(-,+,-)\end{array}\right. \\
\Phi(+,-,-) & = & (+,+,-) \\
\Phi(+,+,-) & = & (+,+,-)
\end{array}\right]$$

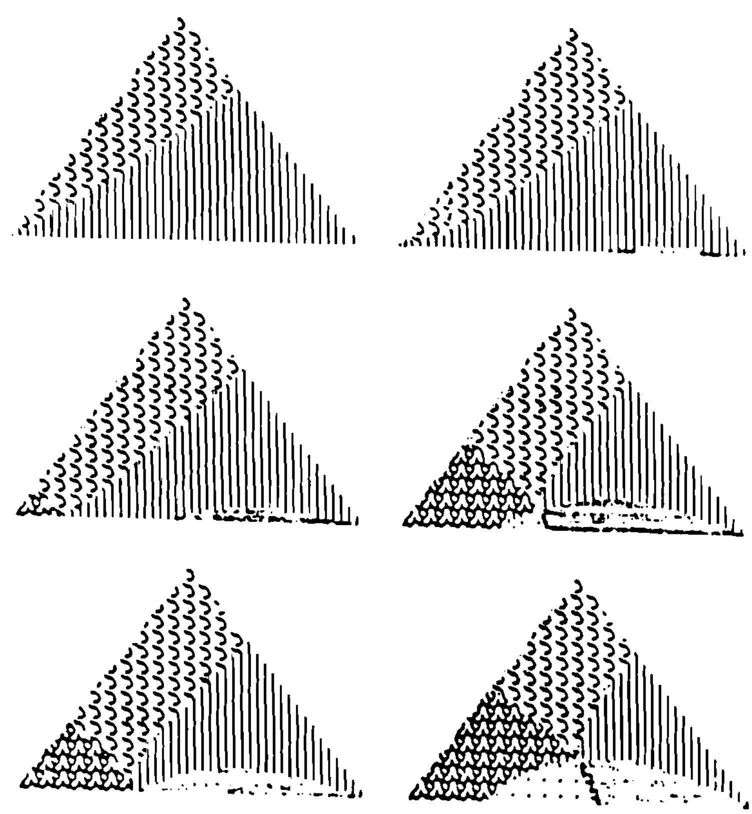
The following qualitative sets are qualitative equilibrium

$$\left\{\begin{array}{l}(-,-,+)\\(+,+,-)\end{array}\right.$$

Computation of the set-valued map Γ

$$\begin{array}{lcl}
\Gamma(-,-,+) & = & (\{-,0\},\{-,0\},\{0\}) \\
\Gamma(-,0,+) & = & (\{0\},\{-,0\},\{0\}) \\
\Gamma(-,+,-) & = & (\{0,+\},\{0\},\{-,0\}) \\
\Gamma(-,+,0) & = & (\{0\},\{0\},\{-,0\}) \\
\Gamma(-,+,+) & = & (\{0\},\{-,0\},\{-,0\}) \\
\Gamma(0,-,+) & = & (\{-,0\},\{0\},\{0\}) \\
\Gamma(0,0,0) & = & (\{0\},\{0\},\{0\}) \\
\Gamma(0,+,-) & = & (\{0,+\},\{0\},\{0\}) \\
\Gamma(+,-,-) & = & (\{0\},\{0,+\},\{-,0,+\}) \\
\Gamma(+,-,0) & = & (\{0\},\{0\},\{-,0,+\}) \\
\Gamma(+,-,+) & = & (\{-,0\},\{0\},\{-,0,+\}) \\
\Gamma(+,0,-) & = & (\{0\},\{0,+\},\{0\}) \\
\Gamma(+,+,-) & = & (\{0,+\},\{0,+\},\{0\})
\end{array}$$

Figure 3: Qualitative Simulation of Replicator Systems # 3



Examples of qualitative cells of replicator systems when the entry $a_{2,1}$ of the matrix

$$A = \begin{pmatrix} 1.00 & 2.00 & -1.00 \\ a_{2,1} & .00 & -2.00 \\ 2.00 & .00 & 1.00 \end{pmatrix}$$

varies from 0 to 3. One can observe cells appearing one after the other and that the transition properties from on cell to the others are preserved.

7 Nonemptiness and Singularity of Qualitative Cells

The question we answer now is whether these qualitative cells are non empty.

Theorem 7.1 *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. Let $\bar{x}$ belong to the qualitative cell K_0. We posit the transversality condition:*

$$g'(\bar{x})C_K(\bar{x}) - a\mathbf{R}^m_+ \ = \ \mathbf{R}^m$$

Then the qualitative cell K_a is nonempty and $\bar{x}$ belongs to its closure. In particular, if

$$g'(\bar{x})C_K(\bar{x}) \ = \ \mathbf{R}^m$$

then the 3^m qualitative cells K_a are nonempty. (We have a pre-chaotic situation *since every qualitative behavior can be implemented as an initial qualitative state.)*

Proof — We apply the Constrained Inverse Function Theorem (see Theorem 4.3.1 of SET-VALUED ANALYSIS) to the map $(x, y) \mapsto g(x) - y$ from $X \times Y$ to Y restricted to the closed subset $K \times a\mathbf{R}^m_+$ at the point $(\bar{x}, 0)$. Its Clarke tangent cone is equal to the product $C_K(\bar{x}) \times a\mathbf{R}^m_+$ since

$$C_{a\mathbf{R}^m_+}(0) \ = \ a\mathbf{R}^m_+$$

Therefore, we know that there exists $\varepsilon > 0$ such that, for all $z \in \varepsilon[-1, +1]^m$, there exist an element $x \in K$ and an element $y \in a\mathbf{R}^m_+$ satisfying $g(x) - y = z$ and $\|x - \bar{x}\| + \|y\| \leq l\|z\|$. Taking in particular $z_i = a_i\varepsilon$, we see that $g(x)_i = a_i\varepsilon + y_i$ and thus, that the sign of $g(x)_i$ is equal to a_i for all $i = 1, \ldots, m$. Hence x belongs to K_a and $\|x - \bar{x}\| \leq l\varepsilon$. □

Let $\bar{x}$ belong to K_0. We shall say that the qualitative cell $\overline{K}_a$ is "singular" at $\bar{x}$ if $\bar{x}$ is locally the only point of the qualitative cell $\overline{K}_a$, i.e., if there exists a neighborhood $N(\bar{x})$ of $\bar{x}$ such that:

$$\forall\, x \in N(\bar{x}) \cap K, \ \ x \neq \bar{x}, \ \ g(x) \ \ \notin \ \ a\mathbf{R}^m_+$$

Theorem 7.2 *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K. Let $\bar{x}$ belong to the qualitative cell K_0.*

We posit the following assumption:

$$T_K(\bar{x}) \cap (g'(\bar{x})^{-1}(a\mathbf{R}_+^m)) = 0$$

Then the qualitative cell $\overline{K}_a$ is singular at $\bar{x}$.

Proof — Assume the contrary: for all $n > 0$, there exists $x_n \in K \cap B(\bar{x}, 1/n)$, $x_n \neq \bar{x}$ such that $g(x_n)$ does belong to $a\mathbf{R}_+^m$. Let us set $h_n := \|x_n - \bar{x}\| > 0$, which converges to 0 and $v_n := \left\|\frac{x_n - \bar{x}}{h_n}\right\|$. Since v_n belongs to the unit ball, which is compact, a subsequence (again denoted) v_n converges to some element v of the unit ball. This limit v belongs also to the contingent cone $T_K(\bar{x})$ because, for all $n > 0$, $\bar{x} + h_n v_n = x_n$ belongs to K.

Finally, since $g(\bar{x} + h_n v_n) = g(x_n) \in a\mathbf{R}_+^m$ for all $n > 0$ and $g(\bar{x}) = 0$, we infer that the limit $g'(\bar{x})v$ of the difference quotients $\frac{g(\bar{x}+h_n v_n) - g(\bar{x})}{h_n} \in a\mathbf{R}_+^m$ belongs to $a\mathbf{R}_+^m$. Hence we have proved the existence of a non zero element

$$v \in T_K(\bar{x}) \cap (g'(\bar{x}))^{-1}(a\mathbf{R}_+^m)$$

a contradiction of the assumption. □

8 General Qualitative Cells

Let us consider the case when K is covered by a finite family $\{K_a\}_{a \in \mathcal{A}}$ of arbitrary closed "qualitative cells" $K_a \subset K$ with nonempty interior:

$$K = \bigcup_{a \in \mathcal{A}} K_a$$

Let $f : K \mapsto X$ be a continuous function with linear growth enjoying the uniqueness property. We denote by $s_f(\cdot)x$ the solution to the differential equation $x' = f(x)$ starting at x when $t = 0$.

It is possible to investigate the qualitative behavior of the system by introducing the following tools:

8.1 Characterization of successors

We denote by

$$\widehat{K} := X\backslash \mathrm{Int}(K) = \overline{X\backslash K}$$

the complement of the interior of K and by

$$\partial K := \overline{K} \cap \widehat{K}$$

the **boundary** of K. We observe that K is the closure of its interior if and only if $X\backslash K$ is the interior of $\widehat{K}$.

We introduce the Dubovitsky-Miliutin cone defined by

Definition 8.1 *The* Dubovitsky-Miliutin tangent cone $D_K(x)$ *to* K *is defined by:*

$$\begin{cases} v \in D_K(x) \text{ if and only if} \\ \exists\ \varepsilon > 0,\ \exists\ \alpha > 0 \text{ such that } x+]0,\alpha](v+\varepsilon B) \subset K \end{cases}$$

Lemma 8.2 *For any x in the boundary of K, the* Dubovitsky-Miliutin cone $D_K(x)$ *to K at x is the complement of the contingent cone $T_{X\backslash K}(x)$ to the complement $X\backslash K$ of K at $x \in \partial K$:*

$$\forall\, x \in \partial K,\ \ D_K(x) = X \setminus T_{X\backslash K}(x)$$

We need the following characterization of the contingent cone to the boundary:

Theorem 8.3 (Quincampoix) *Let K be a closed subset of a normed space and $\widehat{K}$ denote the closure of its complement. Then*

$$\forall\, x \in \partial K,\ \ T_{\partial K}(x) = T_K(x) \cap T_{\widehat{K}}(x)$$

so that the whole space can be partitioned in the following way:

$$\forall\, x \in \partial K,\ \ D_{\mathrm{Int}(K)}(x) \cup D_{X\backslash K}(x) \cup T_{\partial K}(x) = X$$

Proof — If the interior of K is empty, $\partial K = K$, so that the formula holds true. Assume that the interior of K is not empty and take any $x \in \partial K$. Since inclusion $T_{\partial K}(x) \subset T_K(x) \cap T_{\widehat{K}}(x)$ is obviously true, we have to prove that any u in the intersection $T_K(x) \cap T_{\widehat{K}}(x)$ is contingent to the boundary ∂K at x.

Indeed, there exist sequences $k_n > 0$ and $l_n > 0$ converging to $0+$ and sequences $v_n \in X$ and $w_n \in X$ converging to u such that

$$\forall\, n \geq 0, \quad x + k_n v_n \in K \ \ \& \ \ x + l_n w_n \in \widehat{K}$$

We shall prove that there exists $\lambda_n \in [0,1]$ such that, setting

$$h_n \ := \ \lambda_n k_n + (1-\lambda_n) l_n \in [\min(k_n, l_n), \max(k_n, l_n)]$$

and

$$u_n \ := \ \frac{\lambda_n k_n v_n + ((1-\lambda_n) l_n) w_n}{\lambda_n k_n + (1-\lambda_n) l_n}$$

we have

$$\forall\, n \geq 0, \quad x + h_n u_n \in \partial K$$

Indeed, we can take λ_n either 0 or 1 when either $x + k_n v_n$ or $x + l_n w_n$ belongs to the boundary. If not, $x + k_n v_n \in \mathrm{Int}(K)$ and $x + l_n w_n \in X \backslash K$. Since the interval $[0,1]$ is connected, it cannot be covered by the two nonempty disjoint open subsets

$$\Omega_+ \ := \ \{\lambda \in [0,1] \mid x + \lambda k_n v_n + (1-\lambda) l_n w_n \ \in \mathrm{Int}(K)\}$$

and

$$\Omega_- \ := \ \{\lambda \in [0,1] \mid x + \lambda k_n v_n + (1-\lambda) l_n w_n \ \in X \backslash K\}$$

Then there exists $\lambda_n \in [0,1] \backslash (\Omega_+ \cup \Omega_-)$ so that

$$x + \lambda_n k_n v_n + (1-\lambda_n) l_n w_n = x + h_n u_n \ \in \ \partial K$$

Since $h_n > 0$ converges to $0+$ and u_n converges to u, we infer that u belongs to the contingent cone to ∂K.

This formula and Lemma 8.2 imply the decomposition formula. $\square$

We then can split the boundary of ∂K into three areas depending on f:

$$\begin{cases} K_{\Leftarrow} \ := \ \{\, x \in \partial K \mid f(x) \ \in \ D_{\mathrm{Int}(K)}(x) \,\} \\ \qquad \text{the inward area} \\ \\ K_{\Rightarrow} \ := \ \{\, x \in \partial K \mid f(x) \ \in \ D_{X \backslash K}(x) \,\} \\ \qquad \text{the outward area} \\ \\ K_{\Leftrightarrow} \ := \{\, x \in \partial K \ \mid f(x) \in T_{\partial K}(x) \neq \emptyset \,\} \end{cases}$$

Proposition 8.4 *1. — Whenever $x \in K_{\Leftarrow}$, the solution starting at x must enter the interior of K on some open time interval $]0,T[$, and whenever $x \in K_{\Rightarrow}$, the solution starting at x must leave the subset K on some $]0,T[$.*

2. — If $x \in K_{\Leftrightarrow}$, if $\partial K \cap (x + rB) \subset K_{\Leftrightarrow}$ for some $r > 0$ and if f is Lipschitz around x, then the solution starting at x remains in the boundary ∂K on some $[0,T]$.

Proof

1. — Let $x \in K_{\Rightarrow}$ for instance, Then we shall prove that there exist $\rho_x > 0$ and $T_x > 0$ such that

$$\forall\, t \in [0,T_x],\;\; d(s_f(t)x, \partial K) \;\geq\; \rho_x t$$

Indeed, since $f(x) \in D_K(x)$, we associate

$$\rho_x \;:=\; \liminf_{h \to 0+} \frac{d\left(x + hf(x), \widehat{K}\right)}{2h} \;>\; 0$$

This implies that there exists $\tau_x > 0$ such that

$$\forall\, h \in]0,\tau_x],\;\; d(x + hf(x), \widehat{K}) \;\geq\; 2\rho_x h$$

and thus, that

$$\forall\, h \in]0,t_x],\;\; d(x + h(f(x) + \rho_x B), \widehat{K}) \;\geq\; \rho_x h$$

Let us consider now the solution $s_f(\cdot)x$. Since f is continuous, we know that $f(z) \subset f(x) + \rho_x B$ whenever $\|z - x\| \leq \eta_x$ for some η_x.

Since f is bounded by a constant $c > 0$ on the ball $B(x,\eta_x)$, we infer that

$$\|x(t) - x\| \;\leq\; \int_0^t \|f(x(s))\| ds \;\leq\; ct \leq \eta_x$$

when $t \leq T_x := \min\{t_x, \eta_x/c\}$. In this case, we observe that $x(t) - x \in t(f(x) + \rho_x B)$, so that for any $t \in]0,T_x]$,

$$d(x(t), \widehat{K}) \;=\; d(x + x(t) - x, \widehat{K}) \;\geq\; d(x + t(f(x) + \rho_x B), \widehat{K}) \;\geq\; \rho_x t$$

In the same way, we deduce that when $x \in K_{\Leftarrow}$, the solution $s(\cdot)x$ belongs to $X \backslash K$ for $t \in]0, T_x]$.

2. — Take now $x \in K_{\Leftrightarrow}$.

We set $g(t) := d_{\partial K}(x(t))$. Since it is Lipschitz, it is differentiable almost everywhere. Let t be such that $g'(t)$ exists. There exists $\varepsilon(h)$ converging to 0 with h such that

$$x(t+h) \; = \; x(t) + hf(x(t)) + h\varepsilon(h)$$

and

$$g'(t) \; = \; \lim_{h \to 0+} \frac{d_{\partial K}(x(t) + hx'(t) + h\varepsilon(h)) - d_{\partial K}(x(t))}{h}$$

Lemma 5.1.2 of VIABILITY THEORY implies that

$$g'(t) \; \leq \; d(x'(t), T_{\partial K}(\Pi_{\partial K}(x(t))))$$

We denote by $\lambda > 0$ the Lipschitz constant of f and we choose y in $\Pi_{\partial K}(x(t))$. We deduce that

$$d(x'(t), T_{\partial K}(\Pi_{\partial K}(x(t)))) \leq d(x'(t), T_{\partial K}(y)) \leq \|x'(t) - f(y)\|$$
(since $f(y) \in T_{\partial K}(y)$)

$$\leq \|x'(t) - f(x(t))\| + \lambda\|y - x(t)\| \quad \text{(since } f \text{ is Lipschitz)}$$

$$= \; 0 + \lambda d_{\partial K}(x(t)) \; = \; \lambda g(t)$$

Then g is a solution to

$$\text{for almost all } \; t \in [0,T], \;\; g'(t) \; \leq \; \lambda g(t) \;\; \& \;\; g(0) = 0$$

We deduce that $g(t) = 0$ for all $t \in [0,T]$, and thus, that $x(t)$ is viable in ∂K on $[0,T]$. □

As a consequence, we obtain a criterion for a cell to be a successor of another one:

Proposition 8.5 *If $K_b \cap K_c \subset K_{c\Leftarrow}$, then the qualitative cell K_c is a successor of K_b (in the sense that for any $x \in K_b \cap K_c$, there exists τ such that the solution $s(t)x$ remains in K_c for $t \in [0,\tau]$).*

Conversely, if the qualitative cell K_c is a successor of K_b, then

$$K_b \cap K_c \subset K_{c\Leftarrow} \cup K_{c,\Leftrightarrow}$$

8.2 Hitting and Exit Tubes

So far, we have defined the successors of the qualitative cells by the behavior of the dynamical systems on the boundary of the cells.

We shall now investigate what happens to the solutions starting from the interior of the qualitative cells.

For that purpose, we need to introduce the hitting and exit functionals on a continuous function $x(\cdot) \in \mathcal{C}(0,\infty;X)$.

Definition 8.6 *Let $M \subset X$ be a closed subset and $x(\cdot) \in \mathcal{C}(0,\infty;X)$ be a continuous function. We denote by*

$$\varpi_M : \mathcal{C}(0,\infty;X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$$

the hitting functional *associating with $x(\cdot)$ its* hitting time $\varpi_M(x(\cdot))$ *defined by*

$$\varpi_M(x(\cdot)) \ := \ \inf\{t \in [0,+\infty[\mid x(t) \in M\}$$

The function $\varpi_M^\flat : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\varpi_M^\flat(x) \ := \ \varpi_M(s_f(\cdot)x)$$

is called the hitting function. *In the same way, when $K \subset X$ is a closed subset, the functional $\tau_K : \mathcal{C}(0,\infty;X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$ associating with $x(\cdot)$ its* exit time $\tau_K(x(\cdot))$ *defined by*

$$\tau_K(x(\cdot)) \ := \ \inf\{t \in [0,\infty[\mid x(t) \notin K\}$$

is called the exit functional. *the function $\tau_K^\sharp : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by*

$$\tau_K^\sharp(x) \ := \ \tau_K(s_f(\cdot)x)$$

the exit function.

We then note that

$$\varpi_{\widehat{K}}(x(\cdot)) \ \leq \ \tau_K(x(\cdot))$$

that

$$\forall\, t \in [0, \varpi_{\widehat{K}}(x(\cdot))[, \ \ x(t) \in \mathrm{Int}(K) \ \ \& \ \ \forall\, t \in [0, \tau_K(x(\cdot))[, \ \ x(t) \in K$$

and that, when $\varpi_{\widehat{K}}(x(\cdot))$ (respectively $\tau_K(x(\cdot))$) is finite,

$$x(\varpi_{\widehat{K}}(x(\cdot))) \in \partial K \ \ \& \ \ x(\tau_K(x(\cdot))) \in \partial K \ \text{ respectively}$$

Remark also that $\varpi_{\widehat{K}}(x(\cdot)) \equiv 0$ when the interior of K is empty.

We continue to use the convention $\inf\{\emptyset\} := +\infty$, so that $\varpi_{\widehat{K}}(x(\cdot))$ is infinite means that $x(t) \in \mathrm{Int}(K)$ for all $t \in [0, +\infty[$ and that $\tau_K(x(\cdot)) = +\infty$ means that $x(t) \in K$ for all $t \geq 0$.

Lemma 8.7 *Let $K \subset X$ be a closed subset. The functional τ_K and the exit function $\tau_K^\sharp$ are upper semicontinuous when $\mathcal{C}(0, \infty; X)$ is supplied with the pointwise convergence topology. The functional ϖ_M and the hitting function $\varpi_M^\flat$ are lower semicontinuous when $\mathcal{C}(0, \infty; X)$ is supplied with the compact convergence topology.*

Proof — By the Maximum Theorem, the upper semicontinuity of τ_K follows from the lower semicontinuity of the set-valued map $x(\cdot) \rightsquigarrow \Xi(x(\cdot)) \subset \mathbf{R}_+$ where

$$\Xi(x(\cdot)) \ := \ \{t \in [0, \infty[\mid x(t) \notin K\}$$

since $\tau_K(x(\cdot)) = \inf\{\Xi(x(\cdot))\}$.

Indeed, for any $t \in \Xi(x(\cdot))$ and any sequence $x_n(\cdot)$ converging pointwise to $x(\cdot)$, we see that $t \in \Xi(x_n(\cdot))$ for n large enough because $x_n(t)$ belongs to the open set $X \backslash K$ (since $x(t) \in X \backslash K$.)

Let us check now that the function ϖ_M is lower semicontinuous for the compact convergence topology: take any $T \geq 0$ and any sequence $x_n(\cdot)$ satisfying $\varpi_M(x_n(\cdot)) \leq T$ converging to $x(\cdot)$ uniformly over compact subsets and show that $\varpi_M(x(\cdot)) \leq T$. Let us introduce the subsets

$$\Theta_{T'}(x(\cdot)) \ := \ \{t \in [0, T'] \mid x(t) \notin \mathrm{Int}(K)\}$$

By construction, for any $T' > T$, the subsets $\Theta_{T'}(x_n(\cdot))$ are not empty. We also observe that the graph of the set-valued map $x(\cdot) \rightsquigarrow \Theta_{T'}(x(\cdot))$ is closed in the Banach space $\mathcal{C}(0, T'; X) \times [0, T']$: Indeed, if $(x_n(\cdot), t_n) \in \mathrm{Graph}(\Theta_{T'})$ converges to $(x(\cdot), t)$, then $x_n(t_n) \in M$

converges to $x(t)$, which thus belongs to the closed subset M, so that $(x(\cdot), t) \in \text{Graph}(\Theta_{T'})$. Taking its values in the compact interval $[0, T']$, the set-valued map $x(\cdot) \rightsquigarrow \Theta_{T'}(x(\cdot))$ is actually upper semicontinuous. Therefore, for any given $\varepsilon > 0$, $\Theta_{T'}(x_n(\cdot)) \subset \Theta_{T'}(x(\cdot)) + [-\varepsilon, +\varepsilon]$.

We thus infer that $\varpi_M(x(\cdot)) \leq \varpi_M(x_n(\cdot)) + \varepsilon \leq T + \varepsilon$ for every $\varepsilon > 0$. □

We are thus led to single out the following subsets:

Definition 8.8 *We associate with any $T \geq 0$ the subsets*

$$\left\{ \begin{array}{ll} i) & \text{Hit}_f(M,T) := \left\{ x \in X \mid \varpi^{\flat}_M(x) \leq T \right\} \\ \\ ii) & \text{Exit}_f(K,T) := \left\{ x \in K \mid \tau^{\sharp}_K(x) \geq T \right\} \end{array} \right. \tag{8.1}$$

We shall say that the set-valued map $T \rightsquigarrow \text{Hit}_f(M,T)$ is the hitting tube *and that the set-valued map $T \rightsquigarrow \text{Exit}_f(K,T)$ is the* exit tube.

Lemma8.7 implies that the graphs of the hitting and exit tubes are closed.

Proposition 8.9 *Let $K \subset X$ be a closed subset.*

Then $\text{Hit}_f(M,T)$ is the closed subset of initial states x such that closed subset M is reached before T by the solution $s_f(\cdot)x$ to the differential equation starting at x.

The closed subset $\text{Exit}_f(K,T)$ is the subset of initial states $x \in K$ such that the solution $s_f(\cdot)x$ to the differential equation starting at x remains in K for all $t \in [0,T]$. Actually, such a solution satisfies

$$\forall\, t \in [0,T], \;\; s_f(t)x \in \text{Exit}_f(K, T-t)$$

In particular, for $T = +\infty$,

$$\text{Viab}_f(K) \;=\; \text{Exit}_f(K,+\infty) \;=\; \bigcap_{T>0} \text{Exit}_f(K,T)$$

The subset

$$\text{Entr}_f(K) \;:=\; \bigcup_{T>0} \text{Exit}_f(K,T) \;:=\; \left\{ x \in K \mid \tau^{\sharp}_K(x) > 0 \right\}$$

is the subset of elements $x \in K$ from which the solution is viable in K on some nonempty interval $[0,T]$.

We observe that if $T_1 \leq T_2$,

$$\partial K = \mathrm{Hit}_f(K,0) \subset \mathrm{Hit}_f(K,T_1) \subset \mathrm{Hit}_f(K,T_2) \subset \ldots$$

and

$$\mathrm{Viab}_f(K) \subset \mathrm{Exit}_f(K,T_2) \subset \mathrm{Exit}_f(K,T_1) \subset \ldots \subset \mathrm{Entr}_f(K) \subset K$$

Proof — Since the subset of initial states x such that the subset M is reached before T by the solution $x(\cdot)$ to the differential equation starting at x is obviously contained in $\mathrm{Hit}_f(M,T)$, consider an element $x \in \mathrm{Hit}_f(M,T)$ and prove that it satisfies the above property.

By definition of the hitting functional, we can associate a time $t_\varepsilon \leq T + \varepsilon$ such that $x(t_\varepsilon) \in M$.

A subsequence (again denoted by) t_ε converges to $t \in [0, T+\varepsilon]$, so that the limit $x(t)$ of $x(t_\varepsilon) \in M$ belongs to the closed subset M. This implies that $\varpi_M(s_f(\cdot)x) \leq T + \varepsilon$ for every $\varepsilon > 0$.

In the same way, let $T \geq 0$ be finite or infinite. We observe that the subset of initial states $x \in K$ such that a solution $x(\cdot)$ to the differential equation starting at x remains in K for all $t \in [0, T[$ is contained in $\mathrm{Exit}_f(K,T)$, so that it is enough to prove that for any $x \in \mathrm{Exit}_f(K,T)$, the solution $s_f(\cdot)x$ satisfies the above property.

By definition of the exit function, we know that $x(t) \in K$ for any $t \leq \tau_K(s_f(\cdot)x)$ and thus for any $t \leq T$. □

We deduce from Proposition 8.9 a characterization of the successors of a qualitative cell:

Proposition 8.10 *A qualitative cell K_c is a successor of K_b if and only if*

$$K_b \cap K_c \subset \mathrm{Entr}_f(K_c)$$

Let us mention also the following observations¿

Proposition 8.11 *Let $K_a \subset K$ be a closed qualitative cell.*

The complement $K_a \backslash \mathrm{Exit}_f(K_a, T)$ is equal to the set

$$\{x \in K_a \mid \exists\, t \in [0,T] \ \ \text{such that} \ \ s_f(t)x \notin K_a\}$$

of initial states x from which the solution $s_f(\cdot)x$ leaves K_a at some $t \leq T$.

Consequently, if $M \subset K_a \backslash \mathrm{Viab}_f(K_a)$ is compact, there exists $T \geq 0$ such that, for every $x \in M$, there exists $t \in [0,T]$ such that $s_f(t)x \notin K_a$.

In particular, if K_a is a compact repeller, there exists $T < +\infty$ such that for every $x \in K_a$, $s_f(t)x \notin K_a$ for some $t \in [0,T]$.

Proposition 8.9 implies also the following result:

Proposition 8.12 *Let us consider qualitative cells K_a and K_b. Then*

$$K_a \cap \mathrm{Hit}(K_b, T)$$

is the subset of elements of the qualitative cell K_a which reach the qualitative cell K_b before time T and

$$K_a \cap \bigcup_{T \geq 0} \mathrm{Hit}(K_b, T)$$

is the subset of elements of the qualitative cell K_a which reach the qualitative cell K_b in finite time.

Lemma 8.13 *Let us assume that the interior of each qualitative cell is not empty, that they are equal to the closure of their interior and that*

$$\forall\, a \in \mathcal{A}, \ \ \widehat{K_a} \ = \ \bigcup_{b \in \mathcal{A}} K_b$$

Then

$$\forall\, x \in K_a, \ \ \varpi^{\flat}_{\widehat{K_a}}(x) \ = \ \min_{b \in \mathcal{A}} \varpi^{\flat}_{K_b}(x)$$

Therefore, we can cover the qualitative cell K_a by its viability kernel and the closed subcells

$$K_a^b \ := \ \{x \in K_a \mid \tau^{\sharp}_{K_a}(x) \ \geq \ \varpi^{\flat}_{K_b}(x)\}$$

of elements of K_a from which the solution reaches K_b before leaving K_a.

Indeed, either $\tau^\sharp_{K_a}(x)$ is infinite, and x belongs to the viability kernel of the qualitative cell, or it is finite, and thus, there exists at least one qualitative cell K_b such that $\tau^\sharp_{K_a}(x) \geq \varpi^b_{K_b}(x)$, i.e., such that $x \in K^b_a$.

9 Sufficient Conditions for Chaos

Let $f : K \mapsto X$ be a continuous function with linear growth enjoying the uniqueness property. We denote by $s(\cdot)x$ the solution to the differential equation $x' = f(x)$ starting at x when $t = 0$ and by $L(s(\cdot)x)$ its limit set.

Theorem 9.1 *Let us assume that a closed viability domain K of f is covered by a family of compact subsets K_a $(a \in \mathcal{A})$ such that the following "controllability assumption"*

$$\forall\, a \in \mathcal{A}, \forall\, y \in K, \;\; \exists\, x \in K_a, \; t \in [0, \infty[\;\; \textit{such that} \;\; s(t)x = y$$

holds true.

Then, for any sequence $a_0, a_1, \ldots, a_n, \ldots$, there exists at least an initial state $x \in K_{a_0}$ and a nondecreasing sequence of elements $t^j \in [0, \infty]$ such that

$$\begin{cases} i) & s(t^j)x \; \in \; K_{a_j} \; \text{ if } t^j < \infty \\ ii) & L(s(\cdot)x) \cap K_{a_j} \neq \emptyset \; \text{ if } t^j = +\infty \end{cases}$$

The t^j's are finite when we strengthen controllability assumption by assuming that there exists $T \in]0, \infty[$ such that

$$\forall\, a \in \mathcal{A}, \forall\, y \in K, \;\; \exists\, x \in K_a, \; t \in [0, T] \;\; \textit{such that} \;\; s(t)x = y$$

Proof — Let $M \subset K$ be any closed subset. We associate with any $x \in K$ the number $\varpi_M := \inf_{s(t)x \in M} t$, which is nonnegative and finite thanks to the "controllability assumption". We associate with the sequence $a_0, a_1, \ldots$ the subsets $M_{a_0 a_1 \cdots a_n}$ defined by induction by $M_{a_n} := K_n$,

$$M_{a_{n-1} a_n} \; := \; \{x \in K_{a_{n-1}} \mid s(t_{M_{a_n}})x \in K_{a_n}\}$$

and, for $j = n-2, \ldots, 0$, by:

$$M_{a_j a_{j+1} \cdots a_n} := \{x \in K_{a_j} \mid s(t_{M_{a_{j+1} \cdots a_n}})x \in M_{a_{j+1} \cdots a_n}\}$$

They are nonempty closed subsets and form a nonincreasing family. Since K_{a_0} is compact, the intersection $K_\infty := \bigcap_{n=0}^\infty M_{a_0 a_1 \cdots a_n}$ is therefore nonempty.

Let us take an initial state x in K_∞ and fix n. We set $t_n^j := \sum_{k=1}^j \varpi_{M_{a_k \cdots a_n}}$ for any $j = 1, \ldots, n$. We see at once that $s(t_n^j)x \in M_{a_j \cdots a_n} \subset K_{a_j}$.

On the other hand, we observe that $\varpi_{M_2} \leq \varpi_{M_1}$ whenever $M_1 \subset M_2$. Since $M_{a_1 \cdots a_{n+1}} \subset M_{a_1 \cdots a_n}$, we deduce that $t_n^j \leq t_{n+1}^j$ for any $j = 1, \ldots, n$.

Therefore, j being fixed, the nondecreasing sequence t_n^j (for $n \geq j$) converges to some $t^j \in [0, \infty]$. Furthermore, the sequence t^j is not decreasing and, if for some index J, $t^{J-1} < \infty$ and $t^J = \infty$, all the t^j's are equal to $+\infty$ for $j \geq J$.

Since $s(t_n^j)$ belongs to K_{a_j} for all $n \geq j$, we infer that $s(t^j)x$ belongs to K_{a_j} if $j < J$ and that, for $j \geq J$, the intersection $L(s(\cdot)x) \cap K_{a_j}$ is not empty.

If we assume that the stronger assumption holds true, we know that the t_n^j remain in the interval $[0, jT]$, so that the limits t^j are finite. □

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