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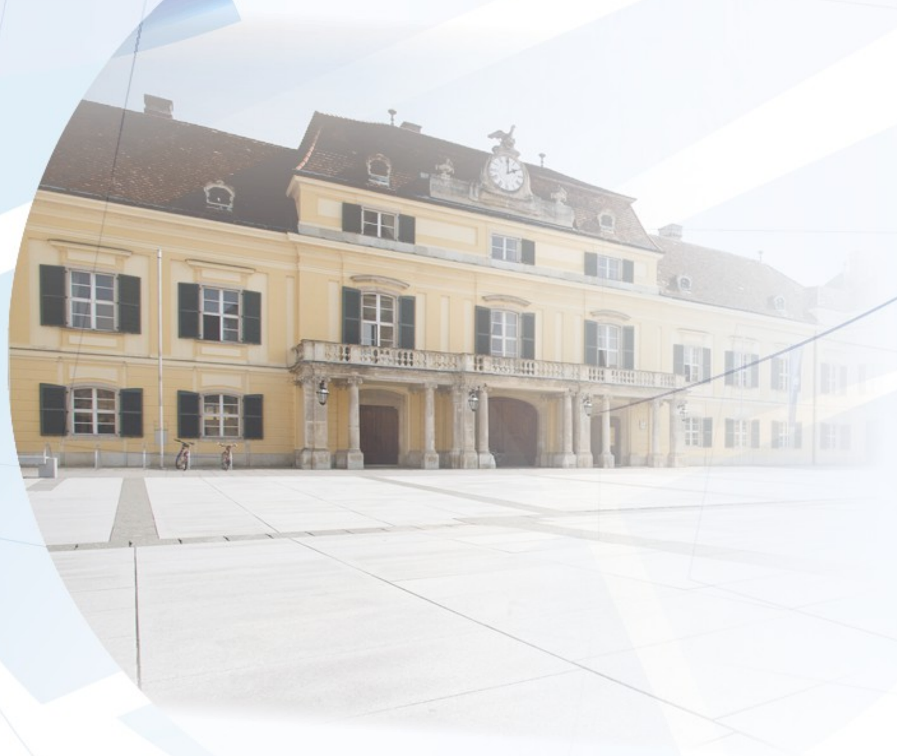

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FOREWORD

A global quantitatiave approach to the study of the stability of the solutions of optimization problems is proposed. It relies on the introduction of a new distance function, namely the hausdorff epigraphical distance.

The authors study the properties of this distance, the underlying topological notions and the corresponding convergence theory for operators.

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CONTENTS

QUANTITATIVE STABILITY OF VARIATIONAL SYSTEMS: I. THE EPIGRAPHICAL DISTANCE

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Abstract
This paper proposes a global measure for the distance between the elements of a variational system (parametrized families of optimization problems).

1. INTRODUCTION AND DEFINITION

The study of the stability of the solutions of optimization problems is a central theme in the optimization literature. It has implication in model formulation, optimality characterizations, approximation theory (especially for infinite dimensional problems), and in particular for numerical procedures. Most of the stability results are topological in nature, i.e., it is shown that under the appropriate conditions the minimum value function, or the set of optimal solutions, possess some type of (semi)continuity. Although there are a few results of a quantitative nature, they are mostly limited to very specific transformations (perturbations) of a restricted class of problems. One of the reasons that no "global" results have been derived, is that there did not seem to exist a good metric, i.e., one with the appropriate theoretical properties and reasonably easy to compute, that could be used to measure the distance between two optimization problems.

In this paper, we study the ***epi-distance***, and show that it has many desirable properties. We then use it, in two subsequent papers [7], [8], to derive hölderian and lipschitzian properties for the optimal, and ϵ-optimal solutions of optimization problems. The framework that serves as background to our study is that of *variational systems* as defined in

Rockafellar and Wets [15], the stress being put on the *global* dependence of optimization problems on parameters that could affect the data that determines the objective as well as the constraints, even the structure of the problem itself.

Although optimization problems, in particular in infinite dimensional spaces, have been our major motivation, one should point out that the results obtained for the epi-distance have also many implications in the convergence theory for operators. This theme is not developed in this paper, but the reader could get an idea of the possibilities from the observations made in Section 2, and in particular from the obvious consequences of Propositions 5.2 and 5.3. Also, the results that we derive here for a functional framework have their immediate counterparts for sets, by specializing them to indicator functions. We illustrate this in just one case. In Section 3, we reformulate Theorem 3.7 in terms of sets. Of course, similar type of corollaries could be worked out for most other theorems and propositions.

After the definition of the epi-distance in Section 1, Section 2 provides a useful criterion for the calculation of the epi-distance in many practical situations. Section 3 makes a comparison between the epi-distance and other notions of distance based on epigraphical regularization (obtained with kernels of the type $(p\lambda)^{-1}\|\cdot\|^p$). Section 4 consists of a few basic observations about the topology induced by the epi-distance, and Section 5 collects some further properties of the epi-distance.

To begin with, let us review some notations and definitions. Unless specifically mentioned otherwise, we always denote by $(X, \|\cdot\|)$ a normed linear space and by d the distance function generated by the norm. For any subset C of X,

$$d(x, C) := \inf_{y\in C}\|x - y\|,$$

denotes the ***distance from x to C***; if $C = \emptyset$ we set $d(x, C) = \infty$. For any $\rho \geq 0$, ρB denotes the ball of radius ρ and for any set C,

$$C_\rho := C \cap \rho B$$

For C, $D \subset X$, the "*excess*" *function* of C on D is defined as,

$$e(C, D) := \sup_{x \in C} d(x, D),$$

with the (natural) convention that $e = 0$ if $C = \emptyset$. Note that the definition implies $e = \infty$ if C is nonempty and D is empty. For any $\rho \geq 0$, the *ρ-(Hausdorff-)distance* between C and D is given by

$$\text{haus}_\rho(C, D) = \sup\{\, e(C_\rho, D),\, e(D_\rho, C)\,\}.$$

DEFINITION 1.1 *For $\rho \geq 0$, the* ρ-(Hausdorff-) epi-distance *between two extended real valued functions f, g defined on X, is*

$$\text{haus}_\rho(f, g) := \text{haus}_\rho(\text{epi}\, f, \text{epi}\, g),$$

where the unit ball of $X \times \mathbf{R}$ is the set $B := B_{X \times \mathbf{R}} = \{(x, \alpha) : \|x\| \leq 1, |\alpha| \leq 1\}$.

One could trace this definition to the one used by Walkup and Wets [19] to measure the distance between convex cones, or that suggested by Mosco [13] to measure the distance between convex sets. But neither one of those earlier papers studies the properties of the epi-distance, or mentions its potential as a tool to obtain quantitative stability (convergence rates).

PROPOSITION 1.2 *Let f_i $(i = 1, 2, 3)$ be extended real valued functions defined on a normed linear space X. For any $\rho \geq 0$,*

(i) nonnegativity: $\text{haus}_\rho(f_1, f_2) \geq 0$;

(ii) symmetry: $\text{haus}_\rho(f_1, f_2) = \text{haus}_\rho(f_2, f_1)$;

(iii) triangle inequality: *for any* $\rho > \inf_{\|x\| \leq \rho} f_i(x)$, $(i = 1,2,3,)$;

$$\text{haus}_\rho(f_1, f_3) \leq \text{haus}_{3\rho}(f_1, f_2) + \text{haus}_{3\rho}(f_2, f_3).$$

Moreover, if f_1 and f_2 are lower semicontinuous, then

(*iv*) *for all* $\rho > 0$, $\text{haus}_\rho(f_1, f_2) = 0$ *if and only* if $f_1 = f_2$.

Note that the condition in (*iii*) is equivalent to $\rho > d(0, (\text{epi}\, f_i)_\rho)$.

PROOF Properties (*i*), (*ii*), and (*iv*) are self-evident. Proving (*iii*) is equivalent to showing that

$$\text{haus}_\rho(C_1, C_3) \leq \text{haus}_{3\rho}(C_1, C_2) + \text{haus}_{3\rho}(C_2, C_3),$$

where $C_i = \text{epi}\, f_i$, $(i = 1, 2, 3)$, are subsets of the normed linear space $(X^\dagger = X \times \mathbb{R}, \|(x, \alpha)\|_\dagger := \max[\|x\|, |\alpha|])$. Let us prove that the above inequality holds with C_1, C_2, C_3 any subsets of a normed linear space Y For $C, D \subset Y$ and $\rho \geq 0$, let

$$\delta_\rho(C, D) = \sup_{\|y\| \leq \rho} |d(y, C) - d(y, D)|,$$

where $\delta_\rho = \infty$ if either C and/or D is empty. Since $\rho B \supset C_\rho$,

$$\delta_\rho(C, D) \geq sup[d(y, D) : y \in C_\rho] = e(C_\rho, D),$$

and hence

$$\delta_\rho(C, D) \geq \text{haus}_\rho(C, D). \tag{1.1}$$

Conversely, for all $\rho > \max[\|y\|, d(0, C)]$,

$$d(y, C) \leq d(0, C) + \rho \leq 2\rho,$$

and thus $d(y, C) = d(y, C_{3\rho})$. It follows that

$$\sup\{d(y, D) - d(y, C) : \|y\| \leq \rho\} \leq \sup\{d(y, D) - d(y, C_{3\rho})\}$$

$$\leq e(C_{3\rho}, D) \ .$$

With the symmetric inequality obtained when interchanging the roles of C and D, this becomes

$$\delta_\rho(C, D) \leq \mathrm{haus}_{3\rho}(C, D) . \tag{1.2}$$

Since δ_ρ clearly satisfies the triangle inequality, (1.1) and (1.2) imply

$$\begin{aligned} \mathrm{haus}_\rho(C_1, C_3) &\leq \delta_\rho(C_1, C_3) \\ &\leq \delta_\rho(C_1, C_2) + \delta_\rho(C_2, C_3) \\ &\leq \mathrm{haus}_{3\rho}(C_1, C_2) + \mathrm{haus}_{3\rho}(C_2, C_3), \end{aligned}$$

provided $\rho > d(0, C_i)$, $(i = 1,2,3)$. □

Rather than defining the epi-distance as done here, one could have considered the Hausdorff distance between the intersection of both epi f and epi g with the ρ-ball, as done in Salinetti and Wets [16]. In general, this distance does not fill our needs, because it does not induce epi-convergence. However in the convex case it would not matter, since it induces the same uniform structure as the epi-distance as we show next. We begin with a couple of preliminary results.

LEMMA 1.3 *Let X be a normed linear space and C a closed convex set such that $C_{\rho_0} \neq \emptyset$. Then for any $\rho > \rho_0$, and $\eta \geq 0$*

$$\mathrm{haus}(C_{\rho+\eta}, C_\rho) \leq [(\rho + \rho_0)/(\rho - \rho_0)] \cdot \eta,$$

which implies that the map $\eta \mapsto \mathrm{haus}(C_{\rho+\eta}, C_\rho)$ is lipschitzian on $\mathbf{R}_+$.

PROOF The argument is based on duality. From Hörmander classical formula, see [5, Section 3] for example,

$$\text{haus}(C_{\rho+\eta}, C_\rho) = \sup\{\,|\,s(C_{\rho+\eta}, x^*) - s(C_\rho, x^*)\,| : \|x^*\| \le 1\}$$

where $s(D, x^*) = \sup\{<x^*, x> : x \in D\}$ is the support function of D. Note that $s(C_\rho, \cdot) = (\delta_C + \delta_{\rho B})^*$, where δ_C is the indicator function of the set C. Moreover $\delta_{\rho B}$ is continuous at a point of the domain of δ_C – because C_{ρ_0} is nonempty and $\rho > \rho_0$ – which means that

$$s(C_\rho, x^*) = \min\{s(C, y^*) + \rho\|x^* - y^*\| : y^* \in X^*\} \tag{1.3}$$

with the minimum attained at some point y_ρ^*. Thus

$$\begin{aligned} &s(C_{\rho+\eta}, x^*) - s(C_\rho, x^*) \\ &\qquad \le \{s(C, y_\rho^*) + (\rho+\eta)\|x^* - y_\rho^*\|\} - \{s(C, y_\rho^*) + \rho\|x^* - y_\rho^*\|\}\ , \\ &\qquad = \eta \cdot \|x^* - y_\rho^*\|\ . \end{aligned}$$

The proof is completed by showing that $\|x^* - y_\rho^*\| \le (\rho + \rho_0)(\rho - \rho_0)^{-1}\|x^*\|$. Indeed

$$\rho\|y_\rho^*\| - \rho\|x^*\| \le \rho\|x^* - y_\rho^*\| \le \rho\|x^*\| + \rho_0\|y_\rho^*\|\ , \tag{1.4}$$

where the last inequality follows from (1.3) with the observations:

$$s(C_\rho, x^*) \le s(\rho B, x^*) \le \rho\|x^*\|\ ,$$

$$s(C, y_\rho^*) \ge <y_\rho^*, x_0> \ge -\rho_0\|y_\rho^*\|\ ,$$

with x_0 any point in C_{ρ_0}. Thus $\|y_\rho^*\| \le 2\rho(\rho - \rho_0)^{-1}\|x^*\|$, which combined with the last relation in (1.4) yields the asserted inequality; recall that $\| x^* \| \leq 1$. □

PROPOSITION 1.4 *Let C and D be two closed subsets of a normed linear space X such that C_{ρ_0} and D_{ρ_0} are nonempty. Then for all $\rho \geq \rho_0$*

$$\mathrm{haus}_\rho(C, D) \leq \mathrm{haus}(C_\rho, D_\rho).$$

Moreover, if C and D are convex, then for $\rho > \rho_0$

$$\mathrm{haus}(C_\rho, D_\rho) \leq \frac{2\rho}{\rho - \rho_0} \mathrm{haus}_\rho(C, D).$$

PROOF The first inequality is self-evident. The second one follows from the "triangle inequality" for the excess function: $e(R, T) \leq e(R, S) + e(S, T)$, for any sets R, S, and T. It implies that

$$\mathrm{haus}(C_\rho, D_\rho) \leq \beta_1 + \beta_2,$$

where for any $\eta > 0$,

$$\beta_1 := \max\left[e(D_\rho, C_{\rho+\eta}), e(C_\rho, D_{\rho+\eta})\right],$$

$$\beta_2 := \max\left[e(C_{\rho+\eta}, C_\rho), e(D_{\rho+\eta}, D_\rho)\right].$$

When C and D are convex, we use Lemma 1.3 to obtain

$$\beta_2 \leq [(\rho + \rho_0)/(\rho - \rho_0)]\eta.$$

With $\eta = \mathrm{haus}_\rho(C, D)$,

$$e(C_\rho, D_{\rho+\eta}) = e(C_\rho, D) \text{ and } e(D_\rho, C_{\rho+\eta}) = e(D_\rho, C),$$

i.e., $\beta_1 = \mathrm{haus}_\rho(C, D)$. This, with the preceding bound for β_2 yields the estimate. □

COROLLARY 1.5 *Let X be a normed linear space and f and g two proper extended real valued lower semicontinuous functions defined on X. Let $\rho_0 > 0$ be such that $(\operatorname{epi} f)_{\rho_0}$ and $(\operatorname{epi} g)_{\rho_0}$ are nonempty. Then for all $\rho \geq \rho_0$,*

$$\operatorname{haus}_\rho(f, g) \leq \operatorname{haus}((\operatorname{epi} f)_\rho, (\operatorname{epi} g)_\rho) .$$

Moreover, if the functions are also convex, for $\rho > \rho_0$

$$\operatorname{haus}((\operatorname{epi} f)_\rho, (\operatorname{epi} g)_\rho) \leq \frac{2\rho}{\rho - \rho_0} \operatorname{haus}_\rho(f, g) .$$

PROOF Simply apply the proposition to the closed epigraphs of f and g. □

2. THE KENMOCHI CONDITIONS

The Kenmochi conditions provide a practical criterion for computing, or at least estimating the epi-distance between two functions.

THEOREM 2.1 *Suppose f, g are proper extended real valued functions defined on a normed linear space X, both minorized by $-\alpha_0 \|\cdot\|^p - \alpha_1$ for some $\alpha_0 \geq 0$, $\alpha_1 \in \mathbf{R}$ and $p \geq 1$. Let $\rho_0 > 0$ be such that $(\operatorname{epi} f)_{\rho_0}$ and $(\operatorname{epi} g)_{\rho_0}$ are nonempty.*

a) *Then the following conditions – to be called the* Kenmochi *conditions – hold: for all $\rho > \rho_0$ and $x \in \operatorname{dom} f$ such that $\|x\| \leq \rho$, $|f(x)| \leq \rho$, for every $\epsilon > 0$ there exists some $\tilde{x}_\epsilon \in \operatorname{dom} g$ that satisfies*

$$\begin{cases} \|x - \tilde{x}_\epsilon\| \leq \operatorname{haus}_\rho(f, g) + \epsilon \\ g(\tilde{x}_\epsilon) \leq f(x) + \operatorname{haus}_\rho(f, g) + \epsilon \end{cases} \tag{2.1}$$

as well as a symmetric condition with the role of f and g interchanged.

b) *Conversely, assuming that for all $\rho > \rho_0 > 0$ there exists a "constant" $\eta(\rho) \in \mathbf{R}_+$, depending on ρ, such that for all $x \in \operatorname{dom} f$ with $\| x \| \leq \rho$, $| f(x) | \leq \rho$, there exists $\tilde{x} \in \operatorname{dom} g$ that satisfies*

$$\begin{cases} \| x - \tilde{x} \| \leq \eta(\rho) \ , \\ g(\tilde{x}) \leq f(x) + \eta(\rho) \ , \end{cases} \tag{2.2}$$

and the symmetric condition (interchanging f and g), then with $\rho_1 := \rho + \alpha_0 \rho^p + \alpha_1$.

$$\operatorname{haus}_\rho(f, g) \leq \eta(\rho_1) . \tag{2.3}$$

PROOF It suffices to observe that

(i) $\operatorname{haus}_\rho(\operatorname{epi} f, \operatorname{epi} g) \leq \theta$, if and only if, for every $\epsilon > 0$

$$(\operatorname{epi} f)_\rho \subset \operatorname{epi} g + (\theta + \epsilon)B \quad \text{and} \quad (\operatorname{epi} g)_\rho \subset \operatorname{epi} f + (\theta + \epsilon)B \ ;$$

(ii) that these inclusions yield exactly the Kenmochi conditions (2.1) if one remembers that epi g is an epigraph;

(iii) the estimate (2.3) is obtained by calculating an upper bound for θ in terms of ρ and $\eta(\rho)$. We do that next.

Given any $(x, \mu) \in (\operatorname{epi} f)_\rho$, i.e., $\| x \| \leq \rho$, $| \mu | \leq \rho$, $\mu \geq f(x)$, we have that $| f(x) | \leq \rho_1$. By (2.2) there exists some $\tilde{x} \in \operatorname{dom} g$ with $\| x - \tilde{x} \| \leq \eta(\rho_1)$ such that

$$g(\tilde{x}) \leq f(x) + \eta(\rho_1) \leq \mu + \eta(\rho_1) .$$

There remains only to observe that if $g(\tilde{x}) \geq \mu$, then $|\mu - \tilde{g}(x)| = \tilde{g}(x) - \mu \leq \eta(\rho_1)$, and

$$d((x,\mu), \operatorname{epi} g) \leq d((x,\mu), (\tilde{x}, g(\tilde{x}))) \leq \eta(\rho_1).$$

On the other hand, if $\mu \geq g(\tilde{x})$, then $(\tilde{x}, \mu) \in \operatorname{epi} g$ and consequently

$$d((x,\mu), \operatorname{epi} g) \leq d((x,\mu), (\tilde{x},\mu)) \leq \eta(\rho_1) \ . \qquad \square$$

REMARK 2.2 Theorem 2.1 tells us that in order to compute $\text{haus}_\rho(f, g)$ we have to find the best constant $\eta(\rho)$ for which the condition (2.2) holds. This condition had been introduced by Kenmochi [11], see also Attouch and Damlamian [4], to study the existence of strong solutions for evolution problems of the following type:

$$0 \in \frac{du}{dt} + \partial f(t, u(t));\ u(0) = u_0 \ .$$

The time dependence of f with respect to t, in our terminology, can now be expressed as an absolute continuity property of the map $t \mapsto f(t)$. It can be formulated as follows: there exist $b \in C([0, T]; H) \cap W^{1,2}([0, T]; H)$ and a, an increasing function, such that:

$$\forall\, 0 \leq s \leq t \leq T, \forall\, x \in \operatorname{dom} f(s, \cdot), \exists\, \tilde{x} \in \operatorname{dom} f(t, \cdot) \text{ such that}$$

$$\|x - \tilde{x}\| \leq |b(t) - b(s)| \cdot (1 + \|x\|) \ ,$$

$$f(t, \tilde{x}) \leq f(s, x) + (a(t) - a(s))(|f(s, x)| + \|x\|^2 + 1) \ .$$

Thus, $\forall\, x \in \operatorname{dom} f(s, \cdot)$ with $\|x\| \leq \rho$, $|f(s, \cdot)| \leq \rho$ we have the existence of some $\tilde{x} \in \operatorname{dom} f(t, \cdot)$ such that

$$\|x - \tilde{x}\| \leq (1 + \rho)\,|b(t) - b(s)| \ ,$$

$$f(t, \tilde{x}) \leq f(s, x) + (1 + \rho + \rho^2)(a(t) - a(s)) \ .$$

Taking $\eta(\rho) = \max\{(1 + \rho)\,|b(t) - b(s)|, (1 + \rho + \rho^2)(a(t) - a(s))\}$, we see that condition (2.2) is satisfied. □

3. COMPARISON WITH THE $d_{\lambda,\rho}$ - DISTANCES

This section is devoted to the relationship between the epi-distance and the distances introduced in Attouch and Wets [5] and [6], based on epigraphical regularizations. Although, one can envisage more general kernels, see Wets [20], Attouch, Azé and Wets [3, Propositions 3.1 and 3.2], for simplicity's sake we shall restrict ourselves to regularizations with respect to *kernels* $k: \mathbf{R}_+ \to \mathbf{R}_+$ of the type:

$$k(r) = \frac{1}{p} r^p \quad \text{for some } p \in [1, \infty) \ .$$

The *epigraphical regularization* f_λ of parameter $\lambda > 0$ of a function $f: X \to \overline{\mathbf{R}}$ (with X a normed linear space) is defined by

$$f_\lambda := f \mathbin{\underset{e}{+}} \lambda^{-1} k(\|\cdot\|)$$

where $\underset{e}{+}$ denotes *epigraphical* sum (inf-convolution):

$$\cdot \ f_\lambda(x) = \inf_{u \in X} \left\{ f(u) + \frac{1}{\lambda p} \|x - u\|^p \right\} \ .$$

With $p = 1$, f_λ is called the Baire-Wijsman approximate, and with $p = 2$, the Moreau-Yosida approximate of f, cf. [5]. Assuming that for some $\alpha > 0$,

$$f + \alpha(\|\cdot\|^p + 1) \geq 0 \ ,$$

then for $0 < \lambda < (\alpha p)^{-1} 2^{-p}$, f_λ is a continuous locally lipschitz function on X, as we show below.

Now fixing the parameter p in the kernel k once and for all, we can define the following distance between two functions f and g

$$d_{\lambda,\rho}(f, g) = \sup_{\|x\| \leq \rho} |f_\lambda(x) - g_\lambda(x)| \ .$$

Assuming that f and g are proper, this quantity is well defined since both f_λ and g_λ are then bounded on bounded sets. These distance functions induce epi-convergence, and in [5, Theorem 2.33 and Corollary 2.42], we obtain quantitative stability results in term of the resolvents of the Moreau-Yosida approximations.

We start with some basic properties of f_λ.

LEMMA 3.1 *Suppose $f \not\equiv \infty$, is an extended real valued function defined on $(X, \|\cdot\|)$ such that for some $\alpha_0 \geq 0$ and $\alpha_1 \in R$, $f + \alpha_0\|\cdot\|^p + \alpha_1 \geq 0$, and $1 \leq p < \infty$. Then for any $0 < \lambda < (\alpha_0 p)^{-1} 2^{1-p}$, f_λ is finite valued. Moreover, for any $x_0 \in X$, $\beta \in \mathbf{R}$*

$$f_\lambda(x + x_0) + \beta = (f(\cdot + x_0) + \beta)_\lambda(x) \ .$$

PROOF The inequality

$$f_\lambda(x) \geq \inf_u \left[- \alpha_0\|x - u\|^p + \frac{1}{p\lambda}\|u\|^p \right] - \alpha_1$$

$$\geq - \alpha_0 2^{p-1}\|x\|^p - \alpha_1$$

follows from

$$\|x - u\|^p \leq 2^{p-1}(\|u\|^p + \|x\|^p) \ ,$$

and $\lambda < (\alpha_0 p)^{-1} 2^{1-p}$. For an upper bound, let x_0 be such that $f(x_0)$ is finite, then

$$f_\lambda(x) \leq f(x_0) + (p\lambda)^{-1}\|x - x_0\|^p \ .$$

Finally,

$$(f(\cdot + x_0) + \beta)_\lambda(x) = \inf_u \left[f(u + x_0) + \beta + \frac{1}{p\lambda}\|x - u\|^p \right]$$

$$= \inf_v \left[f(v) + \frac{1}{p\lambda}\|x + x_0 - v\|^p \right] + \beta \ . \qquad \square$$

The next lemma extends Theorem 2.64 of Attouch [5], proved for Moreau-Yosida approximates, to epigraphical regularizations involving any kernel of the type $(\lambda p)^{-1}\|\cdot\|^p$ for $p \geq 1$.

LEMMA 3.2 *Let $f \not\equiv \infty$ be an extended real valued function defined on $(X, \|\cdot\|)$ such that for some $\alpha_0 \geq 0$, and $\alpha_1 \in \mathbf{R}$, $f + \alpha_0\|\cdot\|^p + \alpha_1 \geq 0$, for given $p \geq 1$. Then for any $0 < \lambda < (\alpha_0 p)^{-1}2^{1-p}$,*

$$f_\lambda = f \underset{e}{+} (p\lambda)^{-1}\|\cdot\|^p$$

is locally lipschitz, i.e.,

$$|f_\lambda(x) - f_\lambda(y)| \leq \lambda^{-1}\kappa\|x - y\| \ ,$$

where the lipschitz constant κ depends continuously on $\|x\|$, $\|x - y\|$, α_0, λ and p; it depends on f only through the value it takes on at some point at which it is finite.

PROOF We have already established that under these assumptions f_λ is finite valued. To simplify the calculations, let us first suppose that $f(0) = 0$. Now from the definition of f_λ, it follows that for all $x \in X$, and $\epsilon > 0$, there exists u_x^ϵ such that

$$f_\lambda(x) \leq f(u_x^\epsilon) + (p\lambda)^{-1}\|x - u_x^\epsilon\|^p \leq f_\lambda(x) + \epsilon \ ,$$

and thus, since f is minorized by $-\alpha_0\|\cdot\|^p - \alpha_1$,

$$-\alpha_0\|u_x^\epsilon\|^p - \alpha_1 + \frac{1}{p\lambda}\|x - u_x^\epsilon\|^p \leq f_\lambda(x) + \epsilon \leq \frac{1}{p\lambda}\|x\|^p + \epsilon \ ,$$

where the last inequality comes from the upper bound

$$f_\lambda(x) \leq f(0) + \frac{1}{p\lambda}\|x - 0\|^p = \frac{1}{p\lambda}\|x\|^p \ .$$

Since $\|u_x^\epsilon\|^p \leq 2^{p-1}(\|x - u_x^\epsilon\|^p + \|x\|^p)$, it follows that

$$\|x - u_x^\epsilon\|^p \left[\frac{1}{p\lambda} - \alpha_0 2^{p-1}\right] \le \left[\frac{1}{p\lambda} + \alpha_0 2^{p-1}\right] \|x\|^p + \alpha_1 + \epsilon$$

With $\alpha' := \alpha_0 p 2^{p-1}$, this yields

$$\|x - u_x^\epsilon\|^p \le \frac{1 + \alpha'\lambda}{1 - \alpha'\lambda} \|x\|^p + \frac{p\lambda}{1 - \alpha'\lambda} (\alpha_1 + \epsilon) \ .$$

For any $y \in X$, we have

$$f_\lambda(y) \le f(u_x^\epsilon) + \frac{1}{p\lambda} \|y - u_x^\epsilon\|^p$$

$$\le f_\lambda(x) + \epsilon + \lambda^{-1} \left[\frac{1}{p} \|y - u_x^\epsilon\|^p - \frac{1}{p} \|x - u_x^\epsilon\|^p\right] \ .$$

We use the convexity of $t \mapsto p^{-1} t^p$ on R_+, and the subgradient inequality to obtain

$$\frac{1}{p} (\|y - x\| + \|x - u_x^\epsilon\|)^p - \frac{1}{p} \|x - u_x^\epsilon\|^p \le (\|y - x\| + \|x - u_x^\epsilon\|)^{p-1} \|x - y\| \ ,$$

and since $\|y - x\| + \|x - u_x^\epsilon\| \ge \|y - u_x^\epsilon\|$, it follows that

$$f_\lambda(y) - f_\lambda(x) \le \epsilon + \lambda^{-1} \|y - x\| (\|y - x\| + \|x - u_x^\epsilon\|)^{p-1} \ .$$

We now use the estimate we have for $\|x - y_x^\epsilon\|$ and let ϵ go to zero. This yields

$$f_\lambda(y) - f_\lambda(x) \le \lambda^{-1} \kappa_x \|y - x\| \ .$$

where

$$\kappa_x = \left\{\|y - x\| + \left[\frac{1 + \alpha'\lambda}{1 - \alpha'\lambda} \|x\|^p + \frac{p\lambda}{1 - \alpha'\lambda} \alpha_1\right]^{1/p}\right\}^{p-1} \ .$$

is a "constant" that depends only on $\|x\|$, $\|y - x\|$, α_0, p, λ. Interchanging the roles of x and y in the above, we obtain a similar inequality with a constant κ_y. Setting $\kappa = \max\ [\kappa_x, \kappa_y]$ yields the desired inequality.

If $f(0) \neq 0$, let x_0 be such that $f(x_0) \in R$. Then $f(\cdot + x_0) - f(x_0)$ is a function that takes on the value 0 at $x = 0$, and moreover, cf. Lemma 3.1,

$$(f(\cdot + x_0) - f(x_0))_\lambda = f_\lambda(\cdot + x_0) - f(x_0) \ .$$

From our earlier argument and this last identity, it follows that

$$|f_\lambda(y) - f_\lambda(x)| \leq \frac{1}{\lambda} \kappa \|y - x\|$$

where in the definition of κ_x the term $\|x\|^p$ is replaced by $\|x - x_0\|^p$ and similarly in κ_y and α_1 is replaced by $\alpha_1 - f(x_0)$. □

LEMMA 3.3 *Let X be a normed linear space, f and g two extended real valued, proper functions defined on X such that for some* $\alpha_0 \geq 0$ *and* $\alpha_1 \in \mathbf{R}$,

$$f + \alpha_0\|\cdot\|^p + \alpha_1 \geq 0, \; g + \alpha_0\|\cdot\|^p + \alpha_1 \geq 0 \ .$$

Then for $1 \leq p < \infty$, *and any* $0 < \lambda < (\alpha_0 p)^{-1} 2^{1-p}$, *and* $\rho \geq \max\,[f_\lambda(0), g_\lambda(0)]$,

$$\mathrm{haus}_\rho(f_\lambda, g_\lambda) \leq \mathrm{haus}_\gamma(f, g) \tag{3.1}$$

where the constant γ, *that depends on* ρ, *is defined by* (3.3).

PROOF There is nothing to prove if $\mathrm{haus}_\gamma(f, g) = \infty$, so let us assume that $\mathrm{haus}_\gamma(f, g)$ is finite; note also that f, g proper implies that epi f and epi g are nonempty, and that $\rho > \max\,[f_\lambda(0), g_\lambda(0)]$ implies that $\mathrm{haus}_\rho(f_\lambda, g_\lambda)$ is finite. To have $\mathrm{haus}_\gamma(f, g) < \eta$ means that

$$(\mathrm{epi}\, f)_\gamma \subset \eta(\mathrm{epi}\, g), \text{ and } (\mathrm{epi}\, g)_\gamma \subset \eta(\mathrm{epi}\, f) \ ,$$

where $\eta D := \{x | d(x, D) \leq \eta\}$ is the η-fattening of D. From this, it follows

$$(\mathrm{epi}\, f)_\gamma + \mathrm{epi}\,(p\lambda)^{-1}\|\cdot\|^p \subset \eta(\mathrm{epi}\, g) + \mathrm{epi}(p\lambda)^{-1}\|\cdot\|^p \ ,$$

and this inclusion, with

$$\text{epi}\, g + \text{epi}(\lambda p)^{-1}\|\cdot\|^p \subset \text{epi}\, g_\lambda$$

yields

$$(\text{epi}\, f)_\gamma + \text{epi}(p\lambda)^{-1}\|\cdot\|^p \subset \eta(\text{epi}\, g_\lambda) \ .$$

Since

$$\text{epi}_s f + \text{epi}_s(p\lambda)^{-1}\|\cdot\|^p = \text{epi}_s f_\lambda \ ,$$

where $\text{epi}_s h := \{(x, \alpha) | \alpha > f(x)\}$ is the *strict epigraph* of h, it suffices to prove that γ can be chosen so that

$$(\text{epi}_s f + \text{epi}_s(\lambda p)^{-1}\|\cdot\|^p)_\rho \subset (\text{epi}\, f)_\gamma + \text{epi}(\lambda p)^{-1}\|\cdot\|^p \ . \tag{3.2}$$

Indeed, the last three identities would imply

$$e((\text{epi}_s f_\lambda)_\rho, \text{epi}\, g_\lambda) \leq \eta \ ,$$

or still, for all $\epsilon > 0$, for all $\eta > \text{haus}_\gamma(f, g)$,

$$e((\text{epi}\, f_\lambda)_{\rho - \epsilon}, \text{epi}\, g_\lambda) \leq \eta + \epsilon \ .$$

The asserted inequality (3.1), now follows from the fact that f_λ is locally lipschitz (Lemma 3.2), and that f and g have symmetric roles.

We turn to (3.2). Let $(x, \alpha) \in (\text{epi}_s f_\lambda)_\rho$. By definition of f_λ, there exists $u_x \in X$ such that

$$\alpha > f(u_x) + (p\lambda)^{-1}\|x - u_x\|^p \ .$$

Moreover, since $(x, \alpha) = (u_x, f(u_x)) + (x - u_x, \alpha - f(u_x))$ and $\alpha - f(u_x) \geq (p\lambda)^{-1}\|x - u_x\|^p$, it suffices to show that there exist γ such that $\|u_x\| \leq \gamma$ and

$|f(u_x)| \leq \gamma$. From $|\alpha| \leq \rho$ and $f + \alpha_0\|\cdot\|^p + \alpha_1 \geq 0$, it follows that

$$- \alpha_0\|u_x\|^p - \alpha_1 + (p\lambda)^{-1}\|x - u_x\|^p \leq \rho$$

The same calculation as in Lemma 3.2 yields

$$\|u_x\|^p \leq (2^{1-p} - \alpha_0\lambda p)^{-1}(\rho^p + p\lambda\rho + p\lambda\alpha_1) := \gamma_1(\rho) \ .$$

From $\alpha > f(u_x)$, we obtain

$$|f(u_x)| \leq \sup\{\rho;\ \alpha_0\|u_x\|^p + \alpha_1\} \ ,$$

and thus we can define γ as

$$\gamma = \gamma(\rho) := \sup\{\rho;\ \gamma_1(\rho)^{1/p};\ \alpha_0\gamma_1(\rho) + \alpha_1\} \ . \tag{3.3}$$

□

THEOREM 3.4 *Let f and g be two extended real valued functions defined on a normed linear space $(X, \|\cdot\|)$, such that for some $\alpha_0 \geq 0$ and $\alpha_1 \in \mathbf{R}$,*

$$f \geq - \alpha_0\|\cdot\|^p - \alpha_1, \text{ and } g \geq - \alpha_0\|\cdot\|^p - \alpha_1 \ ,$$

for $1 \leq p < \infty$. Then for $0 < \lambda < (\alpha_0 p)^{-1}2^{1-p}$, and $\rho \geq 0$

$$d_{\lambda,\rho}(f, g) \leq \beta(\lambda, \rho)\,\text{haus}_{\gamma(\lambda,\rho)}(f, g) \ ,$$

with the constants γ and β as defined in the proof.

PROOF Excluding the cases $f \equiv \infty$ or/and $g \equiv \infty$, when the inequality is trivially satisfied, the functions f_λ and g_λ are finite valued, in fact equi-locally lipschitz, cf. Lemma 3.2. This can be used to conclude that whenever $\|x\| \leq \rho$, both $f_\lambda(x)$ and $g_\lambda(x)$ are bounded in absolute value by

$$\rho' \geq \max\left\{\left|f_\lambda(0) \pm \frac{1}{\lambda}\kappa_\rho \rho\right|, \left|g_\lambda(0) \pm \frac{1}{\lambda}\kappa_\rho \rho\right|\right\} ,$$

where κ_ρ is the lipschitz constant associated to the ρ-ball. (Note that $f_\lambda(0)$ can be bounded by constants that depend only on α_0, α_1, the norm of some point $x^0 \in \operatorname{dom} f$ and $f(x^0)$, and similarly for $g_\lambda(0)$). Setting $\rho_1(\lambda, \rho) = \rho_1 := \max[\rho, \rho']$, let us estimate $g_\lambda(x) - f_\lambda(x)$ when $\|x\| \leq \rho$. By the above, and Kenmochi's conditions (2.1), for all $\epsilon > 0$ there exists y such that $\|x - y\| \leq \text{haus}_{\rho_1}(f_\lambda, g_\lambda) + \epsilon$, and

$$g_\lambda(y) \leq f_\lambda(x) + \text{haus}_{\rho_1}(f_\lambda, g_\lambda) + \epsilon$$

since $\|x\| \leq \rho_1$ and $|f_\lambda(x)| \leq \rho_1$. Because

$$g_\lambda(x) - f_\lambda(x) = (g_\lambda(x) - g_\lambda(y)) + (g_\lambda(y) - f_\lambda(x)) ,$$

it follows that

$$g_\lambda(x) - f_\lambda(x) \leq (p\lambda)^{-1}\kappa_{\rho_2}\|x - y\| + \text{haus}\rho_1(f_\lambda, g_\lambda) + \epsilon$$

$$\leq \beta \text{haus}_{\rho_1}(f_\lambda, g_\lambda) + \beta\epsilon$$

where $\beta = \beta(\lambda, \rho_2) := (p\lambda)^{-1}\kappa_{\rho_2} + 1$ and $\rho_2 := \rho + \text{haus}_{\rho_1}(f_\lambda, g_\lambda) + 1$. With a similar inequality obtained when the role of f and g are interchanged, and after letting ϵ go to 0, this yields

$$d_{\lambda,\rho}(f, g) = \sup_{\|x\| \leq \rho} |f_\lambda(x) - g_\lambda(x)|$$

$$\leq \beta \text{haus}_{\rho_1}(f_\lambda, g_\lambda) \leq \beta \text{haus}_\gamma(f, g) ,$$

where the last inequality follows from Lemma 3.3, and the constant γ is that calculated in the proof of that lemma with ρ_1 replacing ρ in formula (3.3). □

The arguments that we have used in the proof can be viewed as geometric in nature, we give another proof that is of analytic type. It yields a more direct calculation of the lipschitz constant, but does not explicitly bring to the fore the properties of regularized functions (Lemmas 3.1 and 3.2), and the useful inequality of Lemma 3.3.

SECOND PROOF Again, we only need to consider the case when f and g are proper. Pick $x \in X$, $0 < \lambda < (\alpha_0 p)^{-1} 2^{1-p}$, and let us calculate an upper bound for $f_\lambda(x) - g_\lambda(x)$. For $0 < \epsilon \leq 1$, let u_ϵ be such that

$$g(u_\epsilon) + (\lambda p)^{-1} \|x - u_\epsilon\|^p \leq g_\lambda(x) + \epsilon \ ,$$

i.e., u_ϵ attains, up to ϵ, the infimum in the definition of g_λ. Then

$$f_\lambda(x) - g_\lambda(x) \leq \inf_u \{f(u) - g(u_\epsilon) + (\lambda p)^{-1}(\|x - u\|^p - \|x - u_\epsilon\|^p)\} + \epsilon \ . \tag{3.3}$$

Let us begin by deriving an estimate for $\|u_\epsilon\|$. From the assumed minorization of g, it follows that (we denote $\alpha := \max\{\alpha_0, \alpha_1\}$)

$$\begin{aligned} -\alpha(\|u_\epsilon\|^p + 1) + \frac{1}{\lambda p} \|x - u_\epsilon\|^p &\leq g_\lambda(x) + \epsilon \\ &\leq g(u_0) + (\lambda p)^{-1} \|x - u_0\|^p + \epsilon \end{aligned}$$

where u_0 is some arbitrary point in dom g. Hence

$$\begin{aligned} (\lambda p)^{-1} \|x - u_\epsilon\|^p \leq \alpha(1 + 2^{p-1} \|u_\epsilon - x\|^p + 2^{p-1} \|x\|^p) \\ + g(u_0) + (\lambda p)^{-1} \|x - u_0\|^p + \epsilon \ , \end{aligned}$$

$$\|x - u_\epsilon\|^p \leq ((\lambda p)^{-1} - \alpha 2^{p-1})^{-1} [\alpha(1 + 2^{p-1}\|x\|^p) + g(u_0) + (\lambda p)^{-1}\|x - u_0\|^p + \epsilon] \ ,$$

and since $\|u_\epsilon\|^p \leq 2^{p-1}(\|u_\epsilon - x\|^p + \|x\|^p)$, when $\|x\| \leq \rho$:

$$\|u_\epsilon\|^p \leq 2^{p-1}[\rho^p + ((\lambda p)^{-1} - \alpha 2^{p-1})^{-1}[\alpha(1 + 2^{p-1}\rho^p)$$

$$+ g(u_0) + (\lambda p)^{-1}(\|u_0\| + \rho)^p + \epsilon]] \ .$$

This means that $\|u_\epsilon\|$ is bounded above by a constant that depends on ρ, $\|u_0\|$, $g(u_0)$, α, and λ. We are interested in the dependence on λ and ρ, and write

$$\|u_\epsilon\| \leq \gamma_1(\lambda, \rho) \ .$$

Next, we calculate an estimate for $g(u_\epsilon)$. We have

$$g(u_\epsilon) \geq -\alpha(1 + \|u_\epsilon\|^p) \geq -\alpha(1 + \gamma_1(\lambda, \rho)^p) \ .$$

Also

$$g(u_\epsilon) \leq g_\lambda(x) + \epsilon \leq g(u_0) + (\lambda p)^{-1}\|x - u_0\|^p + \epsilon \ .$$

Hence

$$|g(u_\epsilon)| \leq \sup\{\alpha(1 + \gamma_1(\lambda, \rho)^p), g(u_0) + (\lambda p)^{-1}\|x - u_0\|^p + 1\} =: \gamma_2(\lambda, \rho) \ .$$

With $\gamma(\lambda, \rho) := \sup[\gamma_1(\lambda, \rho), \gamma_2(\lambda, \rho)]$, we have that $(u_\epsilon, g(u_\epsilon)) \in B_{X \times \mathbf{R}}(0, \gamma(\lambda, \rho))$. By Theorem 2.1, more precisely by the Kennochi conditions (2.1), we know that there exists v_ϵ such that

$$\|v_\epsilon - u_\epsilon\| \leq \text{haus}_{\gamma(\lambda,\rho)}(f, g) + \epsilon := \eta_\epsilon$$

$$f(v_\epsilon) \leq g(u_\epsilon) + \text{haus}_{\gamma(\lambda,\rho)}(f, g) + \epsilon = g(u_\epsilon) + \eta_\epsilon \ .$$

From (3.3), it follows that

$$f_\lambda(x) - g_\lambda(x) \leq f(v_\epsilon) + (\lambda p)^{-1}\|x - v_\epsilon\|^p - g(u_\epsilon) - (\lambda p)^{-1}\|x - u_\epsilon\|^p + \epsilon \ ,$$

which combined with the preceding inequalities yields

$$f_\lambda(x) - g_\lambda(x) \le \eta_\epsilon + \lambda^{-1}\|x - v_\epsilon\|^{p-1}\|u_\epsilon - v_\epsilon\| + \epsilon \ ,$$

$$\le \eta_\epsilon[1 + \lambda^{-1}(\rho + \gamma(\lambda, \rho) + \eta_\epsilon)^{p-1}] + \epsilon \ .$$

Since this holds for all $\epsilon > 0$, and also when the role of f and g are interchanged, we have

$$|f_\lambda(x) - g_\lambda(x)| \le \text{haus}_{\gamma(\lambda,\rho)}(f, g)[1 + \lambda^{-1}(\rho + \gamma(\lambda, \rho) + \text{haus}_{\gamma(\lambda,\rho)}(f, g)^{p-1})]$$

$$\le \text{haus}_{\gamma(\lambda,\rho)}(f, g)\beta(\lambda, \rho) \ .$$

This completes the proof, since x is an arbitrary point in ρB. □

Our next task is to derive an appropriate bound for haus_ρ in terms of $\text{d}_{\lambda,\rho}$. Again we start with some preparatory lemmas that are of independent interest.

LEMMA 3.5 *Let X be a normed linear space. Suppose f and g are proper, extended real valued functions defined on X. Then for all $\lambda > 0$ and $\rho \ge 0$,*

$$\text{haus}_\rho(f_\lambda, g_\lambda) \le \text{d}_{\lambda,\rho}(f, g).$$

PROOF If $\mu \ge g_\lambda(x)$, $\|x\| \le \rho$, $|\mu| \le \rho$, then $\mu \ge f_\lambda(x) - \text{d}_{\lambda,\rho}(f, g)$. And this implies that $(\mu + \text{d}_{\lambda,\rho}(f, g), x) \in \text{epi}\, f_\lambda$, and thus

$$e((\text{epi}\, g_\lambda)_\rho, \text{epi}\, f_\lambda) \le \text{d}_{\lambda,\rho}(f, g) \ .$$ □

LEMMA 3.6 *Let X be a normed linear space, and f a proper extended real valued function defined on X, minorized by $-\alpha_0\|\cdot\|^p - \alpha_1$ for some $p \ge 1$, $\alpha_0 \ge 0$ and $\alpha_1 \in \mathbf{R}$. Then, for any $0 < \lambda < (\alpha_0 p)^{-1}2^{1-p}$ and $\rho \ge 0$,*

$$\text{haus}_\rho(f_\lambda, f) \le \lambda^{1/p}\left[\frac{p(\alpha_0 2^{p-1}\rho^p + \rho + \alpha_1)}{1 - \alpha_0 p \lambda 2^{p-1}}\right]^{1/p} \ .$$

PROOF Since $f_\lambda \le f$, $e((\text{epi } f)_\rho, \text{epi } f_\lambda) = 0$. To calculate an upper bound for $e((\text{epi } f_\lambda)_\rho, \text{epi } f)$, let $(x, \mu) \in (\text{epi } f_\lambda)_\rho$ - with $\|x\| \le \rho$, $|\mu| \le \rho$ and $\mu \ge f_\lambda(x)$ - and denote by u_x^ϵ an element such that

$$\epsilon + \mu \ge f(u_x^\epsilon) + \frac{1}{p\lambda}\|x - u_x^\epsilon\|^p \ .$$

Note that $(u_x^\epsilon, \epsilon + \mu) \in \text{epi } f$, and thus

$$d((x, \mu), \text{epi } f) \le \max\left[\|x - u_x^\epsilon\|, |\epsilon + \mu - \mu|\right] \ .$$

It thus suffices to obtain a bound on $\|x - u_x^\epsilon\|$. From the minorization of f, and $|\mu| \le \rho$, it follows that

$$\epsilon + \rho \ge - \alpha_0\|u_x^\epsilon\|^p - \alpha_1 + \frac{1}{p\lambda}\|x - u_x^\epsilon\|^p \ .$$

We rely on the inequality $\|u_x\|^p \le 2^{p-1}(\|x\|^p + \|u_x - x\|^p)$, $\|x\| \le \rho$, and $(\lambda p)^{-1} - \alpha_0 2^{p-1} > 0$, to obtain

$$\|x - u_x^\epsilon\|^p \le ((\lambda p)^{-1} - \alpha_0 2^{p-1})^{-1}(\alpha_0 2^{p-1}\rho^p + \rho + \alpha_1) \ ,$$

which yields the asserted bound. □

THEOREM 3.7 *Suppose X is a normed linear space, f and g proper, extended real valued functions defined on X such that for given $p \ge 1$, and some $\alpha_0 \ge 0$, $\alpha_1 \in \mathbf{R}$,*

$$f + \alpha_0\|\cdot\|^p + \alpha_1 \ge 0, \text{ and } g + \alpha_0\|\cdot\|^p + \alpha_1 \ge 0 \ .$$

Then, for all $0 < \lambda < (\alpha_0 p)^{-1}2^{1-p}$, and

$$\rho \ge \max\left[d(0, \text{epi } f), d(0, \text{epi } g)\right] \ ,$$

we have

$$\text{haus}_\rho(f,\, g) \leq d_{\lambda,9\rho}(f,\, g) + 2\lambda^{1/p}\left[\frac{\alpha_0 2^{p-1}(9\rho)^p + 9\rho + \alpha_1}{1 - \alpha_0 p\lambda 2^{p-1}}\right]^{1/p} p^{1/p} \ .$$

PROOF From Proposition 1.2, it follows that

$$\text{haus}(f,\, g) \leq \text{haus}_{9\rho}(f,\, f_\lambda) + \text{haus}_{9\rho}(f_\lambda,\, g_\lambda) + \text{haus}_{9\rho}(g_\lambda,\, g) \ .$$

A direct application of the preceding lemmas yields the upper bound. □

The question of the optimality of the bounds obtained in Theorems 3.4 and 3.7 is important in the derivation of conditioning number to be associated with a nonlinear optimization problem. This is under investigation by Attouch, Azé and Peralba [2].

As mentioned in the Introduction, all the results obtained in this section have their counterpart for sets. However, the constants obtained here may not always be the best ones. Let us consider the case when C, D are nonempty subsets of X. Let $f = \delta_C$ and $g = \delta_D$ be the indicator functions of C and D. Then

$$\text{haus}_\rho(\delta_C,\, \delta_D) = \text{haus}_\rho(C,\, D) \ .$$

Let $p = 1$, then $(\delta_C)_\lambda = \lambda^{-1} d(\cdot,\, C)$, and

$$d_{\lambda,\rho}(C,\, D) = \lambda^{-1} \sup_{\|x\| \leq \rho} |d(x,\, C) - d(x,\, D)|$$

With $\alpha_0 = \alpha_1 = 0$ and

$$\lambda = [(18\rho)^{-1} \sup_{\|x\| \leq 9\rho} |d(x,\, C) - d(x,\, D)|]^{1/2}$$

we derive from Theorem 3.7, the following corollary:

COROLLARY 3.8 *Suppose C, D are nonempty subsets of a normed linear space X. Then*

$$\text{haus}_\rho(C, D) \leq 2[18\rho \sup_{\|x\| \leq 9\rho} |d(x, C) - d(x, D)|]^{1/2} .$$

We conclude this Section by some remarks concerning the distance

$$d^{\#}_{\lambda,\rho}(f, g) = \sup_{\substack{\|x\| \leq \rho \\ \|v\|_* \leq \rho}} |f^{\#}_{\lambda}(x, v) - g^{\#}_{\lambda}(x, v)|$$

where

$$f^{\#}_{\lambda}(x, v) = \inf_{u \in X}\left[f(u) - <v, u> + \frac{1}{p\lambda}\|x - u\|^p\right] + <v, x>$$

$$= \left[(f - <v, \cdot>) \underset{e}{+} \frac{1}{\lambda p}\|\cdot\|^p\right](x) + <v, x> .$$

In |6|, we introduced these distances to extend some of the results obtained in [5] for $d_{\lambda,\rho}$ and X a Hilbert space to the situation when X is a reflexive Banach space. To begin with observe

$$d^{\#}_{\lambda,\rho}(f, g) \geq \sup_{\|x\| \leq \rho} |f^{\#}_{\lambda}(x, 0) - g^{\#}_{\lambda}(x, 0)| = d_{\lambda,\rho}(f, g) .$$

and thus from Theorem 3.7, it follows that with the same conditions on f, g, ρ, and λ,

$$\text{haus}_\rho(f, g) \leq d^{\#}_{\lambda,9\rho}(f, g) + 2\lambda^{1/p}\left[\frac{\alpha'(9\rho)^p + 9\rho + \alpha_1}{1 - p\lambda\alpha'}\right]^{1/p} p^{1/p}$$

where $\alpha' = \alpha_0 2^{p-1}$. On the other hand, since

$$f^{\#}_{\lambda}(x, v) = (f - <v, \cdot>)_\lambda(x) + <v, x> ,$$

it follows that for v fixed, the properties of $f^{\#}_{\lambda}(\cdot, v)$ are essentially the same as those of f_λ, cf. Lemmas 3.1 and 3.2. Moreover, the same arguments as those in the second proof of Theorem 3.4 show that for any $\epsilon > 0$, for $\|x\| \leq \rho$ *and* $\|v\|_* \leq \rho$

$$|f^{\#}_{\lambda}(x, v) - g^{\#}_{\lambda}(x, v)| \leq \eta_\epsilon[1 + \rho + \lambda^{-1}(\rho + \gamma + \eta_\epsilon)^{p-1}]$$

where η_ϵ and $\gamma = \gamma(\lambda, \rho)$ are the same quantities as those that appear in that proof. Hence

$$d^{\#}_{\lambda,\rho}(f, g) \leq \beta^{\#}(\lambda, \rho)\ \mathrm{haus}_{\gamma(\lambda,\rho)}(f, g)$$

with $\beta^{\#}$, the constant β calculated in the proof of Theorem 3.4 plus ρ. We summarize this in the next statement.

THEOREM 3.9 *Suppose X is a normed linear space, f and g proper, extended real valued functions defined on X such that for given $p \geq 1$, and some $\alpha_0 \geq 0$, $\alpha_1 \in \mathbf{R}$,*

$$f + \alpha_0\|\cdot\|^p + \alpha_1 \geq 0, \text{ and } g + \alpha_0\|\cdot\|^p + \alpha_1 \geq 0 \ .$$

Then for $0 < \lambda < (\alpha_0 p)^{-1}2^{1-p}$ and

$$\rho > \max\left[d(0, \mathrm{epi}\, f),\ d(0, \mathrm{epi}\, g)\right] \ ,$$

there exist constants $\beta^{\#}$, γ, κ, that depend on λ and ρ, such that

$$d^{\#}_{\lambda,\rho}(f, g) \leq \beta^{\#}\mathrm{haus}_{\gamma}(f, g) \leq \beta^{\#}\, d^{\#}_{\lambda,9\gamma}(f, g) + \kappa$$

where for fixed ρ, κ can be made arbitrarily small by letting λ go to 0.

4. THE EPI-DISTANCE TOPOLOGY

We limit ourselves to a few basic facts about the topology induced by the pseudo-distances $\{\mathrm{haus}_\rho,\ \rho \geq 0\}$ on the space of extended real functions. Our major concern is its relationship with the topology of epi-convergence. We know that epi-convergence provides the natural conditions, minimal in some sense, under which one can guarantee the convergence of the optimal solutions, see in particular [15, Section 3], [1, Section 2.2].

DEFINITION 4.1 *Let* $\overline{\mathbf{R}}^X$ *be the space of extended real valued functions defined on the normed linear space* X. *The initial topology on* $\overline{\mathbf{R}}^X$ *generated by the pseudo-distances* $\{\text{haus}_\rho,\ \rho \geq 0\}$ – *i.e., the coarsest topology for which the functions* haus_ρ *are continuous – is called the* epi-distance topology. *In other words, for a filtered family* $\{f_\nu,\ \nu \in N\}$

$$f = \text{epi-dist}\lim_\nu f_\nu \ \text{ iff } \lim_\nu \text{haus}_\rho(f_\nu, f) = 0$$

for all $\rho \geq 0$.

Let us begin by observing that the epi-distance topology does only depend on the topology of the underlying space X, not on the specific metric that generates this topology. To be convinced of this, it suffices to return to Section 1, in particular the Definition 1.1, and observe that the excess of a set C_ρ on a set D calculated with a specific norm can always be bounded (below or above) by the excess of C_{ρ_1} on D calculated with another equivalent norm for some $\rho_1 > 0$.

We begin by showing that in finite dimension, a collection of functions epi-converges if and only if it convergences with respect to the epi-distance. Recall that, $\{f^\nu : \mathbf{R}^n \to \overline{\mathbf{R}},\ \nu \in N\}$ a filtered family of functions is said to *epi-converge* to f if for all $x \in \mathbf{R}^n$:

for any collection $\{x^\nu,\ \nu \in N\}$ converging to x $\liminf_\nu f^\nu(x^\nu) \geq f(x)$;

and

there exists $\{x^\nu,\ \nu \in N\}$ converging to x such that $\limsup_\nu f^\nu(x^\nu) \leq f(x)$.

THEOREM 4.2 *Suppose* $X (= \mathbf{R}^n)$ *is finite dimensional. Then, the epi-distance topology is the epi-topology, i.e., the topology of epi-convergence*

The proof is an immediate consequence of the lemma that follows, the reverse implication is immediate. We extend Theorem 2.2v of Salinetti and Wets [17] to the case of a filtered index. Recall that a filtered family $\{f^\nu, \nu \in I\}$ epiconverges to f if

$$\text{epi } f = \limsup_\nu (\text{epi } f^\nu) = \liminf_\nu (\text{epi } f^\nu) \quad ,$$

where for a family $\{C_\nu \subset X, \nu \in I\}$ filtered by $\mathcal{H}$,

$$\limsup_\nu C_\nu = \{x | \forall (Q \in N_{\|\cdot\|}(x), H \in \mathcal{H}), \exists \nu \in H \; s.t. \; C_\nu \cap Q \neq \emptyset\}$$

$$\liminf_\nu C_\nu = \{x | \forall Q \in N_{\|\cdot\|}(x), \exists H \in \mathcal{H} \; s.t. \; \forall \nu \in H, \; C_\nu \cap Q \neq \emptyset\} \quad .$$

LEMMA 4.3 *Let* $X = \mathbf{R}^n$, *and* $\{C_\nu, \nu \in I\}$ *be a family of subsets of* X *filtered by* $\mathcal{H}$. *Then for all* $\rho > 0$,

$$\lim_\nu e((C_\nu)_\rho, (\limsup C_\nu)_\rho) = 0 \quad , \tag{4.1}$$

$$\lim_\nu e((\liminf_\nu C_\nu)_\rho, C_\nu) = 0 \quad . \tag{4.2}$$

If $C = \liminf_\nu C = \limsup_\nu C_\nu$, *then for* $\rho > 0$

$$\lim_\nu \text{haus}_\rho(C_\nu, C) = 0 \quad .$$

PROOF Let $LsC_\nu = \limsup_\nu C_\nu$, and $LiC_\nu = \liminf_\nu C_\nu$. There is nothing to prove if $LsC_\nu = \emptyset$, since then for any $\rho > 0$, there always exists $H \in \mathcal{H}$ such that $(C_\nu)_\rho = \emptyset$ for all $\nu \in H$. Let us thus assume that $LsC_\nu \neq \emptyset$. If (4.1) does not hold, there exist $\epsilon > 0$ and $H \in \mathcal{H}^\#$ (the grill of $\mathcal{H}$) such that for all $\nu \in H$, $e((C_\nu)_\rho, (LsC_\nu)_\rho) > \epsilon$, or equivalently for ν in H, there exists $y^\nu \in (C_\nu)\rho$ such that $d(y^\nu, (LsC_\nu)\rho) > \epsilon$. The collection $\{y^\nu, \nu \in H\} \subset \rho B$ admits at least one cluster point, say $\bar{y} \in \rho B$, which also belongs to

LsC_ν. And for this $\bar{y}$, we have that

$$\lim_{y^\nu \to \bar{y}} d(y^\nu, (LsC_\nu)_\rho) = d(\bar{y}, (LsC_\nu)_\rho) \geq \epsilon > 0$$

which of course, would contradict the fact that $\bar{y} \in (LsC_\nu)_\rho$.

Again if $(LiC_\nu)_\rho = \emptyset$, there is nothing to prove because then $e((LiC_\nu)_\rho, C_\nu) = 0$ whatever be C_ν. Otherwise, simple observe that $(LiC_\nu)_\rho \subset LiC_\nu$, that $e(C_\rho, D) \leq e(C, D)$, and that $\lim e(LiC_\nu, C_\nu) = 0$ as follows from the definition of the lim inf of a collection of sets. □

Let us now turn our attention to the infinite dimensional case, more exactly the case when X is a reflexive Banach space, and epi-limits are defined in terms of Mosco-convergence, i.e. epi-convergence with respect to both the strong and the weak topology on X. Let $\{f^\nu : X \to \overline{\mathbf{R}}, \nu = 1,\ldots\}$ be a sequence of functions. We say that f is the *Mosco-epi-limit* of this sequence, if for all x in X:

$$\text{for any sequence } \{x^\nu, \nu = 1,\ldots\} \text{ converging weakly to } x, \liminf_\nu f^\nu(x^\nu) \geq f(x) \quad ,$$

and

$$\text{there exists } \{x^\nu, \nu = 1,\ldots\} \text{ converging strongly to } x \text{ such that } \limsup_\nu f^\nu(x^\nu) \leq f(x) \ .$$

Since in infinite dimensions, every Mosco-epi-limit is necessarily weakly lower semicontinuous, we are naturally led to focus our attention to the subspace of convex functions. It is then rather easy to see that the convergence of the epi-distances implies Mosco-epi-convergence. Actually, in this setting, the epi-distance topology is strictly finer than the Mosco-epi-topology. We demonstrate all of this in what follows. Also that in the context provided by the important applications of epi-convergence in infinite dimensional, whenever a sequence Mosco-epi-converges to f it also converges with respect to the epi-distance topology.

To begin with let us record an important consequence of Theorems 3.4 and 3.7 and 3.9.

THEOREM 4.4 *The topology induced on the space of functions* $\overline{\mathbf{R}}^X$ *defined on the normed linear space* X *by the pseudo-distances* $\{d_{\lambda,\rho};\ \lambda > 0, \rho \geq 0\}$, *or* $\{d^{\#}_{\lambda,\rho};\ \lambda > 0, \rho \geq 0\}$ *is the epi-distance topology.*

In the Hilbert case, we know of one more collection of pseudo-distances $\{d^{J}_{\lambda,\rho};\ \lambda > 0, \rho \geq 0\}$ that induces the same topology on the space of proper lower semicontinuous functions on X. This follows from the preceding theorem and [5, Theorem 2.33]. The distance $d^{J}_{\lambda,\rho}$ is computed as the supremum on ρ-balls of the distance between the resolvents of the Moreau-Yosida approximates of parameter λ. This equivalence is exploited in the proofs of Propositions 5.2 and 5.3.

In view of this, there appears to be two important topologies that can be defined on the space of proper lower semicontinuous convex functions defined on a reflexive Banach space: the Mosco-epi-topology and the epi-distance topology. The question of knowing if they are equivalent goes begging. One verifies readily that the Mosco-epi-topology is coarser. The example below shows that it is strictly coarser.

PROPOSITON 4.5 *Suppose* X *is a reflexive Banach space,* $\{f;\ f^{\nu}, \nu = 1,\dots\}$ *a collection of proper, extended real valued, lower semicontinuous, convex functions defined on* X. *Then,*

$$\lim_{\nu \to \infty} \text{haus}_{\rho}(f, f^{\nu}) = 0 \ ,$$

for all ρ *sufficiently large implies*

$$f = \text{Mosco-epi-}\lim_{\nu \to \infty} f^{\nu} \ .$$

PROOF Simply use Theorem 3.4 combined with Theorem 8 of [6]. □

EXAMPLE 4.6 *Let X be the Hilbert space* $L^2(\Omega, \mathbf{R})$,

$$f^\nu(u) = \frac{1}{2} \int_\Omega a_\nu(x)\ u^2(x)dx, \quad \nu = 1,\ldots$$

$$f(u) = \frac{1}{2} \int_\Omega a(x)\ u^2(x)dx\ .$$

We consider the Moreau-Yosida approximates,

$$f^\nu_\lambda(u) = \left[f \underset{e}{+} \frac{1}{2\lambda}\ \|\cdot\|^2\right](u)$$

$$= \frac{1}{2} \int_\Omega a_\nu \frac{u^2}{(1 + \lambda a_\nu)^2} dx + \frac{1}{2\lambda} \int_\Omega u^2 \left[1 - \frac{1}{1 + \lambda a_\nu}\right]^2 dx$$

$$= \frac{1}{2} \int_\Omega \frac{a_\nu}{1 + \lambda a_\nu} u^2 dx\ ,$$

hence

$$d_{\lambda,1}(f^\nu, f) = \sup_{\|u\|_{L^2} \le 1} |f^\nu_\lambda(u) - f_\lambda(u)|$$

$$= \sup_{\|v\|_{L^1} \le 1} \int_\Omega \left| \frac{a}{1 + \lambda a} - \frac{a_\nu}{1 + \lambda a_\nu} \right| v dx$$

$$= \left\| \frac{a}{1 + \lambda a} - \frac{a_\nu}{1 + \lambda a_\nu} \right\|_{L^\infty}\ .$$

Now, take $\Omega = [0, 1]$, $a_\nu(x) = x^{1/\nu}$ and $a(x) = 1$. Then

$$\left\| \frac{1}{1 + \lambda} - \frac{1}{x^{-1/\nu} + \lambda} \right\|_\infty = \frac{1}{1 + \lambda} = d_{\lambda,1}(f^\nu, f)$$

that does not go to 0. Thus, the f^ν does not converge in the epi-distance topology to f.

But they do Mosco-epi-converge. Simply observe that for all $\lambda > 0$, the sequence $\{f_\lambda^\nu, \nu = 1,\dots\}$ is increasing, and pointwise converges to f_λ, which implies Mosco-epi-convergence, see [1, Theorem 3.26]. □

However, usually one is in the situation covered by the next theorem.

THEOREM 4.7 *Suppose X and H are two Hilbert spaces and $X \hookrightarrow H$ is a continuous compact embedding. Then for any collection $\{f;\, f^\nu, \nu = 1,\dots\}$ of proper, equi-coercive, lower semicontinuous convex functions defined on X, the following statements are equivalent*

(i) $f = \text{Mosco-epi-}\lim_{\nu \to \infty} f^\nu$ on H;

(ii) *for all ρ sufficiently large,* $\lim_{\nu \to \infty} \text{haus}_\rho(f, f^\nu) = 0$;

where the epi-distance is defined in terms of the norm on H.

(The collection $\{f^\nu, \nu \in N\}$ is *equi-coercive* if there exists a function $\theta : \mathbb{R}_+ \to [0, \infty)$ with $\lim_{t \to \infty} \theta(t) = \infty$ such that for all $\nu \in N$, $f^\nu(x) \geq \theta(\|x\|)$ for all $x \in X$.)

PROOF Follows from [1, Theorem 2.55] and Theorems 3.4 and 3.7. □

We note that the distances $d_{\lambda,\rho}$ and $d^{\#}_{\lambda,\rho}$ have been defined here in terms of epigraphical regularizations obtained by taking the epigraphical sum with a polynomial kernel, one can reasonably conjecture that distances defined by epigraphical regularization with respect to a much wider class of kernels are going to be equivalent to the epi-distance. A complete description would be useful; it is still an open question.

By relying on the relation between $d^{\#}_{\lambda,\rho}(f, g)$ and $d^{\#}_{\lambda,\rho}(f^*, g^*)$ derived in [6, Theorem 5], the next result is obtained as an immediate consequence of Theorem 3.9.

THEOREM 4.8 *The Legendre-Fenchel transform,*

$$f \mapsto f^* : \Gamma_0(X) \to \Gamma_0(X^*)$$

is continuous for the epi-distance topology, where X is a reflexive Banach space, and $\Gamma_0(X)$ [resp. $\Gamma_0(X^*)$] is the space of proper, extended real valued, lower semicontinuous functions defined on X [resp. X^*].

Of course, the epi-distance topology is metrizable. Simply use the pseudo-distances to construct the metric. The next result shows that it is also complete, under some restrictions. That this is also the case in general is an open question.

PROPOSITION 4.9 *The space of extended real valued functions defined on a normed linear space X equipped with the uniform structure generated by the epi-distances $\{\text{haus}_\rho;\ \rho \geq 0\}$ is complete, in the two following situations:*

(i) *X is a finite dimensional space*

(ii) *X is a reflexive Banach space and the functions are proper, lower semicontinuous and convex.*

PROOF Let $\{f_n;\ n \in \mathbb{N}\}$ be a Cauchy sequence, i.e., for all $\rho > 0$,

$$\text{haus}_\rho(f_n, f_m) \to 0 \text{ as } n \text{ and } m \text{ go to } \infty \ .$$

From Theorem 3.4 and 3.7, this is equivalent to:

$$\forall \rho > 0 \ \ \forall \lambda > 0 \ \ d_{\lambda,\rho}(f_n, f_m) \to 0 \text{ as } n,\ m \text{ go to } \infty \ ,$$

where

$$d_{\lambda,\rho}(f_n, f_m) = \sup_{\|x\| \leq \rho} |(f_n)_\lambda(x) - (f_m)_\lambda(x)|$$

and $(f_n)_\lambda$ is computed for some kernel k of the form $k(\cdot) = \frac{1}{p}\|\cdot\|^p$. We choose $k = \frac{1}{2}\|\cdot\|^2$ to simplify the calculations. Hence for every $\rho \geq 0$ and $\lambda > 0$ $\{(f_n)_\lambda;\ n \in \mathbb{N}\}$ is on ρB, a Cauchy sequence for the distance of uniform convergence. Therefore for every $\lambda > 0$, there exists some function f^λ such that for all $\rho \geq 0$,

$(f_n)_\lambda \to f^\lambda$ uniformly on ρB .

The difficulty is to show that the *family* $\{f^\lambda;\ \lambda > 0\}$ *is the epigraphical regularization of a given function* f. If such a function exists it is necessarily given by the following formula

$$f := \sup_{\lambda > 0} f^\lambda \ .$$

Let us compute $(f)_\mu = f \dotplus \frac{1}{2\mu} \|\cdot\|^2$ for $\mu > 0$ and prove that in case i) or ii) the following equality holds

$$(f)_\mu = f^\mu$$

which will clearly imply the assertion. We first observe that given any extended real valued function g on X, X being only assumed to be a normed linear space, the epigraphical regularizations of g for various indices are connected by the so-called resolvent equation [1], see [5, (2.5)] for a proof that also applies here,

$$\text{for all } \lambda, \mu > 0,\ ((g)_\lambda)_\mu = (g)_{\lambda + \mu} \ .$$

We apply this with $g = f_n$, and pass to the limit as n goes to ∞. Noticing that

$$((f_n)_\lambda)_\mu(x) = \inf_{u \in X}\left[(f_n)_\lambda(u) + \frac{1}{2\mu}\|x - u\|^2\right]$$

$$= \inf_{u \in \rho_0 B}\left[(f_n)_\lambda(u) + \frac{1}{2\mu}\|x - u\|^2\right]$$

for some $\rho_0 \geq 0$ independent of n, as follows from the uniform convergence of $(f_n)_\lambda$ to f^λ on any bounded ball in X, we have that

$$\text{for all } x,\ (f^\lambda)_\mu(x) = \lim_{n \to \infty} ((f_n)_\lambda)_\mu(x) \ .$$

Since

$$((f_n)_\lambda)_\mu = (f_n)_{\lambda+\mu} \to f^{\lambda+\mu} \text{ as } n \to \infty \ ,$$

we can conclude that for all λ, $\mu > 0$

$$(f^\lambda)_\mu = f^{\lambda+\mu} \ .$$

Given $\mu > 0$, let us take the supremum with respect to $\lambda > 0$ in this formula. Clearly $\theta \to f^\theta(x)$ is an increasing locally lipschitz function from $\mathbf{R}^+$ into $\mathbf{R}$. Hence

$$\sup_{\lambda>0} (f^\lambda)_\mu = f^\mu \ .$$

The only thing we need to prove in order to complete the proof is that

$$\sup_{\lambda>0} (f^\lambda)_\mu = f_\mu \ .$$

Observing that f^λ increases to f as $\lambda \downarrow 0$, we are in the following situation:

Given $f_n \uparrow f$ does

$$\inf_{u\in X}\left\{f_n(u) + \frac{1}{2\mu}\|x - u\|^2\right\} \uparrow \inf_{u\in X}\left\{f(u) + \frac{1}{2\mu}\|x - u\|^2\right\} \ ?$$

This is clearly verified in situations (i) and (ii). In case (ii) just note that the sequence $\{f_n;\ n \in \mathbf{N}\}$ Mosco-epi-converges to f and that the set of minimizers of the above expressions is clearly bounded and thus relatively weakly compact. □

5. FURTHER PROPERTIES

We complete our study with two propositions. The first one that follows from the results of Azé and Penot [9], and the second that relies on the interplay between haus_ρ and $d_{\lambda,\rho}$, are here to serve as examples of the tools that are available for calculating the epi-distance.

PROPOSITION 5.1 [9] *Let X be a Banach space, $(f_i, i = 1,\dots n)$ and $(g_i, i = 1,\dots,n)$ proper, lower semicontinuous, convex functions defined on X with values in $\mathbf{R} \cup \{\infty\}$. Assume that these functions are minorized by $-\alpha(\|\cdot\|^p + 1)$ for some $p \geq 1$ and $\alpha \geq 0$, and*

$$(\sigma B)^n \subset \text{diag } X^n \cap (\gamma B)^n - \Pi_{i=1}^n (\text{lev}_\gamma f_i)_\gamma$$

for some $\gamma \geq 0$ and $\sigma > 0$ where

$$\text{diag } X^n := \text{diag } \Pi_{i=1}^n X_{(i)}, \text{ here each } X_{(i)} = X \quad ,$$

$$\text{lev}_\gamma f_i := \{x : f_i(x) \leq \gamma\}$$

Then, for all $\rho \geq n\gamma + \sigma$, assuming that $\sum_{i=1}^n \text{haus}_{\rho_1}(f_i, g_i) < \sigma$,

$$\text{haus}_\rho\left[\sum_{i=1}^n f_i, \sum_{i=1}^n g_i\right] \leq \frac{n\gamma + \sigma + \rho}{\sigma} \sum_{i=1}^n \text{haus}_{\rho_1}(f_i, g_i) \quad ,$$

where $\rho_1 = \rho + (n+1)[\alpha(\rho + \sigma + 1) + \sigma]$.

Because of its properties, in particular the characterization provided by the Kenmochi conditions (2.1), the epi-distance is relatively easy to calculate or to estimate in most applications. On the other hand, the distances $d_{\lambda,\rho}$ and $d^{\#}_{\lambda,\rho}$ based on epigraphical regularization are better suited for theoretical investigations; for example, one can demonstrate that the Legendre-Fenchel transform is an isometry for those distances [5]. One of the major consequences of Theorems 3.4, 3.7 and 3.9, is that they give us the flexibility to use either one in our calculation. The proof of our next result illustrates this point.

PROPOSITION 5.2 *Suppose X is a Hilbert space, f and g, proper, extended real valued, lower semicontinuous convex functions defined on X. To any $\rho > \max[d(0, \text{epi} f),\ d(0, \text{epi} g)]$, there corresponds constants γ, and κ (that depend on ρ) such that*

$$\text{haus}_\rho(gph\, \partial f,\ gph\, \partial g) \leq \kappa[\text{haus}_\gamma(f, g)]^{1/2}$$

where for an operator $A : X \rightrightarrows X$,

$$gph\, A = \{(x, y) | y \in A(x)\} \ ,$$

is the graph of the operator A.

PROOF The idea is to use as intermediate result, one that comes from $d_{1,\rho}$ for Moreau-Yosida approximates of f and g:

$$f_\lambda = f \mathbin{\dot{+}} \frac{1}{2} \|\cdot\|^2,\ g_\lambda = g \mathbin{\dot{+}} \frac{1}{2} \|\cdot\|^2 \ .$$

The kernel $\frac{1}{2} \|\cdot\|^2$ is particularly well adapted to the Hilbert space setting.

The convexity and properness of f and the coercivity of $\|\cdot\|^2$ guarantee the existence of a unique point $J_1^f(x) = (I + \partial f)^{-1}(x)$, called the *resolvent* (of parameter 1) at x such that

$$J_1^f(x) = \text{argmin}_u \left[f(u) + \frac{1}{2} \|x - u\|^2 \right] \ .$$

(The function $x \mapsto J_1^f(x)$ is also called the *proximal map.*) Now, observe that $(x, y) \in gph\, \partial f$, implies that

$$(x + y) \in (I + \partial f)(x) \ ,$$

and hence

$$x = (I + \partial f)^{-1}(x + y) = J_1^f(x + y)$$

$$y = (x + y) - J_1^f(x + y)$$

With $z = x + y$, this yields

$$gph\,\partial f = \{(J_1^f(z),\, z - J_1^f(z)) : z \in H\} \ .$$

Since J_1^f is a contraction, it follows that $gph\,\partial f$ is a lipschitzian manifold, cf. Brezis [10], Rockafellar [14]. In particular

$$(gph\,\partial f)_\rho \subset \{(J_1^f(z),\, z - J_1^f(z)) : \|z\| \leq 2\rho\} \ .$$

Similarly

$$(gph\,\partial g) = \{(J_1^g(z),\, z - J_1^g(z)) : z \in H\} \ .$$

And thus

$$e((gph\,\partial f)_\rho,\, gph\,\partial g) \leq \sup_{\|z\| \leq 2\rho} \|J_1^f(z) - J_1^g(z)\| \ .$$

Theorem 2.33 of [5], gives us the inequality

$$d_{1,2\rho}^J(f, g) := \sup_{\|z\| \leq 2\rho} \|J_1^f(z) - J_1^g(z)\| \leq \kappa'[d_{1,\gamma'}(f, g)]^{1/2}$$

where $\kappa' = 2\sqrt{2}$, and $\gamma' = 4\rho + \|J_1^f(0)\| + \|J_1^g(0)\|$. And in turn this, with Theorem 3.4, yields

$$e((gph\,\partial f)_\rho,\, gph\,\partial g) \leq \kappa[\text{haus}_\gamma(f, g)]^{1/2} \ ,$$

where the constants κ and γ depend on κ' and γ' and the same quantities than those that appear in the calculation in the proof of Theorem 3.4; we note that because f and g are proper convex functions there always exist $\alpha_0 \geq 0$ and $\alpha_1 \in \mathbf{R}$ such that $-\alpha_0\|\cdot\|^2 - \alpha_1$

minorizes f and g. □

REMARK 5.3 This last theorem improves a result of Schultz [18] obtained when $X = R^n$, and f and g are the sum of two continuous convex functions with the same indicator function of a closed set. Also, note that the *exponent* 1/2 *is optimal.* Simply consider $X = \mathbf{R}$, $f(t) = \frac{\alpha}{2}|t|$, and $g(t) = \frac{\alpha}{2}|t - \alpha|$ for some $\alpha > 0$. Then for $\rho > 0$, $\text{haus}_\rho(f, g) = \frac{1}{2}\alpha^2$ and $\text{haus}_\rho(gph\,\partial f, gph\,\partial g) = \alpha$.

PROPOSITION 5.4 *Suppose X is a Hilbert space, f and g are proper, extended real valued, lower semicontinuous, convex functions defined on X. Then, for any $\rho > 0$ and $\lambda > 0$,*

$$d_{\lambda,\rho}(f, g) \leq (2 + \lambda)\lambda^{-1}\rho\,\text{haus}_\gamma(gph\,\partial f, gph\,\partial g) + \alpha_\lambda(f, g) \quad .$$

where

$$\gamma = \gamma(\lambda, \rho) := \sup\{\|J_\lambda^f 0\| + \rho;\ \lambda^{-1}(2\rho + \|J_\lambda^f 0\|)\} \quad ,$$

$$\alpha_\lambda(f, g) := |f_\lambda(0) - g_\lambda(0)| \quad .$$

PROOF With the same notations as in Proposition 5.2, let us start from the inequality [5, Proposition 2.30],

$$d_{\lambda,\rho}(f, g) \leq \lambda^{-1}\rho d_{\lambda,\rho}^J(f, g) + \alpha_\lambda(f, g) \quad ,$$

where $\alpha_\lambda(f, g)$ is defined above, and

$$d_{\lambda,\rho}^J(f, g) = \sup_{\|x\| \leq \rho}\|J_\lambda^f x - J_\lambda^g x\| \quad ,$$

$$J_\lambda^f x = \text{argmin}_u\left[f(u) + \frac{1}{2\lambda}\|x - u\|^2\right] \quad .$$

The remainder of the proof shows that $d^J_{\lambda,\rho}(f, g) \leq (2 + \lambda)\text{haus}_\gamma(gph\,\partial f, gph\,\partial g)$.

From the optimality conditions for $J^f_\lambda x$ in the expression above, if necessary see [10, p. 39] for details,

$$(J^f_\lambda x,\ \lambda^{-1}(x - J^f_\lambda x)) \in gph\,\partial f\ .$$

Moreover, assuming that $||x|| \leq \rho$, from the contraction properties of J^f_λ, it follows that

$$||J^f_\lambda x|| \leq ||J^f_\lambda 0|| + \rho$$

$$||\lambda^{-1}(x - J^f_\lambda x)|| \leq \lambda^{-1}|\rho + ||J^f_\lambda 0|| + \rho|$$

Hence

$$(J^f_\lambda x,\ \lambda^{-1}(x - J^f_\lambda x)) \in (gph\,\partial f)_\gamma$$

with γ as defined above. If $\text{haus}_\gamma(gph\,\partial f, gph\,\partial g) \leq \eta$, then for all $\epsilon > 0$, there exists $(y^\epsilon, v^\epsilon) \in gph\,\partial g$ such that

$$||y^\epsilon - J^f_\lambda x|| \leq \eta + \epsilon\ ,$$

$$||v^\epsilon - \lambda^{-1}(x - J^f_\lambda x)|| \leq \eta + \epsilon\ .$$

When $u^\epsilon = y^\epsilon + \lambda v^\epsilon$, cf. the proof of Proposition 5.2,

$$y^\epsilon = J^g_\lambda u^\epsilon,\ v^\epsilon = \lambda^{-1}(u_\epsilon - J^g_\lambda u^\epsilon)\ ,$$

we have

$$||J^f_\lambda x - J^g_\lambda u^\epsilon|| \leq \eta + \epsilon$$

$$\lambda^{-1}||(u^\epsilon - x) - (J^g_\lambda u^\epsilon - J^f_\lambda x)|| \leq \eta + \epsilon\ .$$

The last two inequalities imply

$$\|u^\epsilon - x\| \leq \|J^g_\lambda u^\epsilon - J^f_\lambda x\| + \lambda(\eta + \epsilon) \leq (1 + \lambda)(\eta + \epsilon) \ .$$

Let us now use the triangle inequality,

$$\|J^f_\lambda x - J^g_\lambda x\| \leq \|J^f_\lambda x - J^g_\lambda u^\epsilon\| + \|J^g_\lambda u^\epsilon - J^g_\lambda x\| \ ,$$

the contraction property of J^g_λ to bound $\|J^g_\lambda u^\epsilon - J^g_\lambda x\|$, and the bounds on $\|u^\epsilon - x\|$, and $\|J^f_\lambda x - J^g_\lambda u^\epsilon\|$, to conclude

$$\|J^f_\lambda x - J^g_\lambda x\| \leq (2 + \lambda)(\eta + \epsilon) \ .$$

Letting ϵ go to zero, and taking the supremum over ρB yields $d^J_{\lambda,\rho}(f, g) \leq (2 + \lambda)\eta$, and completes the proof. □

COROLLARY 5.5 *Under the same assumptions as in* Proposition 5.3, *and with* $\alpha_0 \geq 0$ *and* $\alpha_1 \in \mathbf{R}$ *such that* $-\alpha_0\|\cdot\|^2 - \alpha_1$ *minorizes* f *and* g, *then for all* $0 < \lambda < (4\alpha_0)^{-1}$ *and* $\rho >$ *max* $[d(0, \text{epi} f), d(0, \text{epi} g)]$,

$$\text{haus}_\rho(f, g) \leq 9(2 + \lambda)\lambda^{-1}\rho\, \text{haus}_\gamma(gph\, \partial f, gph\, \partial g) + \kappa$$

where $\kappa = \alpha_\lambda(f, g) + 4\sqrt{\lambda}\,(162\alpha_0\rho^2 + 9\rho + \alpha_1)^{1/2}(1 - 4\alpha_0\lambda)^{-1/2}$, *and* α_λ *and* $\gamma = \gamma(\lambda, 9\rho)$ *are the constants defined in* Proposition 5.3.

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