



International Institute for  
Applied Systems Analysis  
[www.iiasa.ac.at](http://www.iiasa.ac.at)

# Some Further Optimality Conditions for the CM-Embedded Problem

Xia, Z.Q., Strodiot, J.-J. & Nguyen, V.H.

IIASA Working Paper

WP-88-051

July 1988



Xia ZQ, Strodiot J-J, & Nguyen VH (1988). Some Further Optimality Conditions for the CM-Embedded Problem. IIASA Working Paper. IIASA, Laxenburg, Austria: WP-88-051 Copyright © 1988 by the author(s).  
<http://pure.iiasa.ac.at/id/eprint/3154/>

**Working Papers** on work of the International Institute for Applied Systems Analysis receive only limited review. Views or opinions expressed herein do not necessarily represent those of the Institute, its National Member Organizations, or other organizations supporting the work. All rights reserved. Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage. All copies must bear this notice and the full citation on the first page. For other purposes, to republish, to post on servers or to redistribute to lists, permission must be sought by contacting [repository@iiasa.ac.at](mailto:repository@iiasa.ac.at)

# ***WORKING PAPER***

## **SOME FURTHER OPTIMALITY CONDITIONS FOR THE $C_M$ -EMBEDDED PROBLEM**

*Z.Q. Xia  
J.-J. Strodiot  
V. Hien Nguyen*

July 1988  
WP-88-051

**SOME FURTHER OPTIMALITY CONDITIONS  
FOR THE  $C_M$ -EMBEDDED PROBLEM**

*Z.Q. Xia  
J.-J. Strodiot  
V. Hien Nguyen*

July 1988  
WP-88-051

*Working Papers* are interim reports on work of the International Institute for Applied Systems Analysis and have received only limited review. Views or opinions expressed herein do not necessarily represent those of the Institute or of its National Member Organizations.

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS  
A-2361 Laxenburg, Austria

## **AUTHORS**

Professor Z.Q. Xia is from the Department of Applied Mathematics, Dalian Institute of Technology, Dalian, China.

Professor J.-J. Strodiot and Professor V. Hien Nguyen are from the Dept. of Mathematics, Facultés Universitaires N.-D. de la Paix, Rempart de la Vierge 8, B-5000 Namur, Belgium.

## FOREWORD

Normally, executing an algorithm for solving a  $C_M$ -embedded problem is stopped at a point where some necessary conditions are satisfied. However, for the  $C_M$ -embedded problem both necessary and sufficient conditions may be found. This paper is contributed to explore new optimality conditions some of which are both necessary and sufficient conditions. They could be used to verify if a solution generated by an algorithm is at least a locally optimal solution to the  $C_M$ -embedded problem, and used to construct ascent algorithms for this problem with non-convex regions in practical calculations.

Alexander Kurzhanski  
Chairman  
System and Decision Sciences Program

## ABSTRACT

In this paper some new optimality conditions for the  $C_M$ -embedded problem with the Euclidean norm are presented. Some of them are both necessary and sufficient for certain non-convex regions. The results associated with optimality conditions given here could be used to construct ascent algorithms and in practical calculations.

**Keywords.** Design centering, lineality cone, quasi-differentiable functions,  $C_M$ -embedded problem.

# SOME FURTHER OPTIMALITY CONDITIONS FOR THE $C_M$ -EMBEDDED PROBLEM

*Z.Q. Xia, J.-J. Strodiot and V. Hien Nguyen*

## INTRODUCTION

We consider the  $C_M$ -embedded problem as follows:

$$\max_{x \in S} \min_{\xi \in \Omega} \min_{y \in D_\xi} \|x - y\|^2, \quad (1.1)$$

where  $\text{int}S \neq \emptyset$  and

$$S = \{x \in \mathbb{R}^n \mid f_\xi(x) \leq f_\xi^0, \xi \in \Omega\}$$

being bounded and simply connected and satisfying  $\nabla f_i(y) \neq 0$ ,  $y \in \text{bd}S$ ,  $i \in \Omega$ ,  $\Omega$  is a finite index set, and

$$D_\xi = \{y \in \mathbb{R}^n \mid f_\xi(y) \geq f_\xi^0, f_\eta(y) \leq f_\eta^0, \eta \in \Omega \setminus \{\xi\}\}, \xi \in \Omega.$$

There are different ways to describe this problem, for instance,

$$\begin{aligned} & \max_{r, x} r \\ & \text{s.t. } f_i(x + wr) \leq f_i^0, \quad i \in \Omega, \quad w \in B(0, 1), \end{aligned}$$

[1]-[5], [10]. The function

$$\varphi(x) = \min_{\xi \in \Omega} \min_{y \in D_\xi} \|x - y\|^2$$

is Lipschitzian and quasidifferentiable in the sense of [7]. Some optimality conditions concerned have been proposed, [1], [2], [5] - [10]. In the case where nonconvex regions are determined by convex and complementary convex constraints, say, the generalized Fritz-John necessary conditions,

$$0 \in \lambda_0 e_{n+1} + \sum_{i \in J_g} \lambda_i \partial \gamma_i(c^*, r^*) + \sum_{j \in J_h} \lambda_j \partial \eta_j(c^*, r^*) \quad (1.2)$$

$$\lambda_0, \lambda_i \geq 0, \sum_0^m \lambda_i \neq 0, m = J_g + J_h,$$

$$\lambda_i(\gamma_i - g_i^0) = 0, i \in J_g,$$

$$\lambda_j(\eta_j - h_j^0) = 0, j \in J_h,$$

where  $(c^*, r^*)$  is an optimal solution to the problem

$$\max_{c, r} r$$

$$s.t. \gamma_i(c, r) \leq g_i^0, i \in J_g,$$

$$\eta_j(c, r) \leq h_j^0, j \in J_h$$

where

$$\gamma(c, r) = \max_{w \in B(0,1)} g(c+rw), \eta(c, r) = \max_{w \in B(0,1)} h(c+rw),$$

and a sufficient condition have been proposed, due to [1]. Under the same constraint condition a necessary condition different from (1.2) could be deduced from [13], [14] and  $\varphi(\cdot)^{1/2}$  is a d.c. function, [14]. In [9] some concrete necessary conditions and expansible directions were also described. In [10] a sufficient condition for general nonconvex regions was given. As for the case where all of the constraints which determine a feasible region are quasiconvex, or quasiconcave (complementary convex), the necessary and sufficient conditions could be found in some literatures, e.g., [6], [7]. The necessary conditions for this problem, e.g., (1.2), that have been proposed are not very efficient to be used in the practical computation. The sufficient conditions that were presented and proved in [1], [8] and [10], are more efficient to be used in the practical computation and in constructing numerical methods. But it seems that for the concrete problem, the  $C_M$ -embedded problem, it is possible to find both necessary and sufficient conditions in order to expect to provide new bases on which more efficient search techniques could be constructed.

This paper is contributed to explore new optimality conditions which are both necessary and sufficient conditions. This is the basic purpose of this paper.

It should be mentioned that some notations given in [10], and also used here, are listed below:

- $\psi(x, \xi, y) = \|x - y\|^2, \forall y \in D_\xi, \xi \in \Omega, x \in S.$

- $Y = \times_{\Omega} D_{\xi} \subset \mathbf{R}^{n \times m}$ .
- $Y(x, \xi) = \{y \in \mathbf{R}^n \mid \psi(x, \xi, y) = \min_{y \in D_{\xi}} \psi(x, \xi, y)\}, \xi \in \Omega$ .
- $\Omega(x) = \{\xi \in \Omega \mid \min_{y \in D_{\xi}} \psi(x, \xi, y) = \min_{\xi \in \Omega} \min_{y \in D_{\xi}} \psi(x, \xi, y)\}, x \in S$ .
- $Y(x) = \bigcup_{\xi \in \Omega(x)} Y(x, \xi)$ .
- $\partial\varphi(x)$  is the quasidifferential of  $\varphi$  at  $x$  in the sense of [7].
- $\mathbf{CA}$  is the conical hull of the set  $A$ .
- $L_{\mathbf{CA}}$  is the lineality space in the cone  $\mathbf{CA}$ .
- $G(x; Q) = \{x - y \mid y \in Q \subset \mathbf{R}^n\} = x - Q$ . Normally, we suppose  $Q \subset Y(x)$ .
- $N^+(x; Q) = \{h \in \mathbf{R}^n \mid \langle u, h \rangle \geq 0, u \in G(x; Q)\}$ , or if  $Q = Y(x)$ , then  
 $N^+(x; Y(x)) = \{h \in \mathbf{R}^n \mid \langle u, h \rangle \geq 0, u \in \partial\varphi(x)\}$ .

## 1. SOME FURTHER OPTIMALITY CONDITIONS

Some notations will be introduced below in order to further develop optimality conditions to the problem (1.1). The notation

$$A \underline{SP}(y) B$$

means that sets  $A$  and  $B$  can be separated by some hyperplane  $H(y)$  at the point  $y$ . In our present discussion, the hyperplane  $H(y)$  is just the tangent hyperplane  $Tf_{\xi}(y)$  at the point  $y \in \text{bd } S$ , i.e., satisfying

$$\|y - x\|^2 = \varphi(x), \tag{2.1}$$

$$f_{\xi}(y) = f_{\xi}^0,$$

$$f_{\eta}(y) \leq f_{\eta}^0, \eta \in \Omega \setminus \{\xi\}$$

for some  $\xi \in \Omega(x)$ .  $U_{\epsilon(y)}(y)$  denotes a neighborhood at  $y$ , mostly an Euclidean ball with radius  $\epsilon(y)$ . Define

$$\overset{\vee}{P}(x) := \{y \in Y(x) \mid \exists \epsilon(y) > 0 : B(x, \varphi(x)^{1/2}) \underline{SP}(y) U_{\epsilon(y)}(y) \cap \text{bd } S\}.$$

A point  $x$  is said to be a  $\vee$ -point if  $x \in \text{int } S$  and  $\overset{\vee}{P}(x) = Y(x)$ . A point  $y \in \overset{\vee}{P}(x)$  is said to be an open-type point associated with  $x$ . An open-type point is said to be strict if there exists  $\epsilon(y) > 0$  such that

$$U_{\epsilon(y)}(y) \cap bd S \cap H(y) = \{y\} . \quad (2.2)$$

A point  $x$  is called a strict  $\vee$ -point if every point  $y \in \hat{P}(x)$  is strict.

Similarly, we can define the opposite type point. The symbol

$$A \underline{SS}(y) B$$

means that sets  $A$  and  $B$  are in the same closed half-space determined by a supporting hyperplane  $H(y)$  of  $A$  or  $B$  at  $y \in A \cap B$ . Define

$$\hat{P}(x) := \{y \in Y(x) \mid \exists \epsilon(y) > 0 : B(x, \varphi(x)^{1/2}) \underline{SS}(y) U_{\epsilon(y)}(y) \cap bd S \text{ and} \\ B(x, \varphi(x)^{1/2}) \cap bd S \cap U_{\epsilon(y)}(y) = \{y\}\} .$$

As mentioned above, the supporting hyperplane  $H(y)$  is just the tangent hyperplane  $Tf_{\xi}(y)$  at  $y \in bd S$  satisfying the condition (2.1). A point  $x$  is said to be a  $\wedge$ -point if  $x \in int S$  and  $\hat{P}(x) = Y(x)$ . Each point  $y \in \hat{P}(x)$  is called a closed-type point associated with  $x$ . A point  $y \in \hat{P}(x)$  is called strict if it satisfies the condition (2.2). A point  $x$  is called a strict  $\wedge$ -point if every point  $y \in \hat{P}(y)$  is strict. A point  $y \in \hat{P}(x)$  is called trivial if there exists  $\epsilon(y) > 0$  such that

$$U_{\epsilon(y)}(y) \cap bd S \subset Tf_{\xi}(y)$$

for some  $\xi \in \Omega(x)$ . In the definition of  $\hat{P}(x)$ , we excluded the case in which for some point  $y \in Y(x)$  and, for all  $\epsilon > 0$ , one has

$$U_{\epsilon}(y) \cap bd S \cap B(x, \varphi(x)^{1/2}) \setminus \{y\} \neq \emptyset \text{ and} \quad (2.3) \\ B(x, \varphi(x)^{1/2}) \underline{SS}(y) U_{\epsilon}(y) \cap bd S .$$

The set consisting of the points, in  $Y(x)$ , satisfying (2.3) is denoted by  $\bar{P}(x)$ . Similarly, we can define strictness for the points in  $\bar{P}(x)$  and  $x \in int S$ .

The other situation is defined as follows. Define

$$\tilde{P}(x) := \{y \in Y(x) \mid \forall \epsilon > 0 : U_{\epsilon}(y) \cap bd S \cap H(y; \nabla f_{\xi}(y), >) \neq \emptyset \text{ and}$$

$$U_{\epsilon}(y) \cap bd S \cap H(y; \nabla f_{\xi}(y), <) \neq \emptyset \text{ for some } \xi \in \Omega(x) \text{ satisfying (2.1)}\}$$

where  $H(y; \nabla f_{\xi}(y), >)$  denotes an open half-space, i.e.,

$$H(y; \nabla f_{\xi}(y), >) = \{y' \in \mathbf{R}^n \mid \langle y' - y, \nabla f_{\xi}(y) \rangle > 0\}$$

and  $H(y; \nabla f_{\xi}(y), <)$  denotes the other open half-space, i.e.,

$$H(y; \nabla f_{\xi}(y), \langle \cdot \rangle) = \{y' \in \mathbb{R}^n \mid \langle y' - y, \nabla f_{\xi}(y) \rangle < 0\} .$$

A point  $y \in \tilde{P}(x)$  is said to be a general type point associated with  $x$ .

**Theorem 2.1.** Suppose that  $x$  is a strict  $\vee$ -point. If the point  $x \in \text{int } S$  is a locally optimal solution to the problem (1.1), then  $N^+(x; Y(x))$  is only the singleton containing the zero-vector, i.e.,

$$\dim N^+(x; Y(x)) = 0 . \quad (2.4)$$

**Proof.** For the sake of contradiction, suppose that  $\dim N^+(x; Y(x)) \neq 0$ . From the opposite assumption there exists such an  $h \in \mathbb{R}^n$  that  $h \in N^+(x; Y(x))$  and  $h \neq 0$ . Since  $x$  is a locally optimal solution, there exists a  $\delta_0 > 0$  such that

$$\varphi(x + \lambda h) - \varphi(x) \leq 0 \quad (2.5)$$

for all  $\lambda \in (0, \delta_0)$ . Now we prove that there exists a  $\delta > 0$  such that  $\varphi(x + \lambda h) - \varphi(x) > 0$  for all  $\lambda \in (0, \delta)$ , that is, for all  $\lambda \in (0, \delta)$ , one has

$$d^2(x + \lambda h, bd S) > \varphi(x) . \quad (2.6)$$

Suppose that (2.6) is not true. Then there exist sequences  $\{\delta_i\}_1^\infty \downarrow 0$  and  $\{\lambda_i\}_1^\infty$  such that  $0 < \lambda_i \leq \delta_i$  and

$$d^2(x + \lambda_i h, bd S) \leq \varphi(x), \forall i . \quad (2.7)$$

A sequence of points  $\{z_i\} \subset bd S$  can be found such that, for any  $i$ ,

$$d^2(x + \lambda_i h, z_i) = d^2(x + \lambda_i h, bd S) \leq \varphi(x) . \quad (2.8)$$

Since  $\Omega$  is finite and  $Y(x)$  is compact, there exist  $\bar{\xi} \in \Omega$  and a subsequence  $\{i_k\}_1^\infty \subset \{i\}_1^\infty$  such that the corresponding subsequence  $\{z_{i_k}\}$  converges to a point  $\bar{y}$  of the boundary of  $S$  and the whole subsequence  $\{z_{i_k}\}$  is on the  $(bd S)_{\bar{\xi}}$  where  $(bd S)_{\bar{\xi}}$  denotes  $S \cap D_{\bar{\xi}}$ . Since  $(bd S)_{\bar{\xi}}$  is compact,  $\bar{y} \in (bd S)_{\bar{\xi}}$ . In consequence of the continuity of the function  $d(\cdot, \cdot)$  and (2.8), one has

$$\begin{aligned} d^2(x + \lambda_{i_k} h, z_{i_k}) &= d^2(x + \lambda_{i_k} h, (bd S)_{\bar{\xi}}) \xrightarrow{\text{as } k \rightarrow \infty} d^2(x, \bar{y}) \\ &= d^2(x, (bd S)_{\bar{\xi}}) = d^2(x, bd S) = \varphi(x) \end{aligned} \quad (2.9)$$

that is,  $\bar{y} \in Y(x)$ . In view of  $h \in N^+(x; Y(x))$ , we have

$$\lambda_{i_k} h \in N^+(x; Y(x)) \text{ or } \langle x - \bar{y}, \lambda_{i_k} h \rangle \geq 0, \forall i_k .$$

We showed that  $\{x + \lambda_{i_k} h\}_1^\infty$  is included in the closed half-space  $H^+(x; x - \bar{y})$ , i.e.,

$$\{x + \lambda_{i_k} h\}_1^\infty \subset \{w \mid \langle x - \bar{y}, w - x \rangle \geq 0\} .$$

However the hyperplane  $\{w \mid \langle x - \bar{y}, w - x \rangle = 0\}$  is parallel to the tangent hyperplane  $Tf_\xi(\bar{y})$ . Since  $x$  is strict, it is impossible for (2.8) to hold. This contradiction shows that (2.6) is true. But (2.6) contradicts (2.5). This contradiction shows that  $N^+(x; Y(x)) = \{0\}$ , i.e.,  $\dim N^+(x; Y(x)) = 0$ . The proof of this theorem is completed.  $\square$

**Remark 2.2.** If we only assume that  $x$  is a  $v$ -point, then (2.4) is still a necessary condition for a strictly locally optimal solution to the problem (1.1).

From [10, Theorem 2.4] and Theorem 2.9 we are able to get a necessary and sufficient condition to a locally optimal solution of the problem (1.1) in the case where the center is a  $v$ -point.

### Optimality Condition 2.3

Suppose that  $x$  is a  $v$ -point and strict.  $x$  is a locally optimal solution to the problem (1.1) if and only if

$$\dim N^+(x; Y(x)) = 0 \tag{2.10}$$

or

$$\dim L_{CG\partial\varphi(x)} = n . \tag{2.11}$$

**Remark 2.4.** Suppose that  $Y(x) = \hat{P}(x) \cup \bigvee \hat{P}(x)$  and furthermore that

$$L_{CG(x; Y(x))} \cap G(x; \hat{P}(x)) = \{\emptyset\} \tag{2.12}$$

Then  $x$  is not a locally optimal solution, and (2.12) holds if and only if

$$N^+(x; \bigvee \hat{P}(x)) \cap \text{int } N^+(x; \hat{P}(x)) \neq \emptyset . \tag{2.13}$$

**Lemma 2.5.** If  $x$  is a locally optimal solution to the problem (1.1), then  $\dim L_{CG\partial\varphi(x)} \neq 0$ , i.e.,  $\dim N^+(x; Y(x)) \neq n$ .

**Proof.** Suppose that  $\dim L_{CG(x; Y(x))} = 0$ . From Theorem 2.1, there exists a non-zero direction  $h \in \text{int } N^+(x; Y(x))$ . Here, for all  $u \in G(x; Y(x))$ ,  $\langle u, h \rangle > 0$ . It follows immediately that

$$\langle u, h \rangle > 0 \quad \forall u \in \partial \varphi(x) .$$

Because  $\partial \varphi(x)$  is compact and  $\partial \varphi(x) = \text{co} \{2u \mid u \in G(x; Y(x))\}$ , one has

$$\varphi(x; h) = \min_{u \in \partial \varphi(x)} \langle u, h \rangle > 0 .$$

Thus  $B(x, \varphi(x)^{1/2})$  is able to expand continuously along the direction  $h$ , and for all  $\epsilon > 0$ , there exists  $z \in U_\epsilon(x)$  such that  $\varphi(z) > \varphi(x)$ . This contradicts the fact that  $x$  is a locally optimal solution.  $\square$

**Theorem 2.6.** If the condition

$$\dim L_{CG(x; \hat{P}(x))} \neq 0 \tag{2.14}$$

holds, then  $x$  is a locally optimal solution to the problem (1.1).

**Proof.** Let

$$\dim L_{CG(x; \hat{P}(x))} = k > 0 .$$

For all  $y \in \hat{P}(x)$ , there exists  $\delta(y) > 0$  and  $\epsilon(y)$  satisfying  $0 < \epsilon(y) < \delta(y)$  such that

$$B(x, \varphi(x)^{1/2}) \underline{SS}(y) U_{\delta(y)}(y) \cap bd S$$

and, for any  $z \in [U_{\epsilon(y)}(x) \cap \{z \mid \langle x-y, z-x \rangle = 0\}]$ , one has

$$d(z, U_{\delta(y)}(y) \cap bd S) \leq \varphi(x)^{1/2} . \tag{2.15}$$

In fact, let  $u := Proj z / Tf(y)$ .  $w$  denotes such a point that  $w \in U_{\delta(y)}(y) \cap bd S$  and

$$d(z, w) = d(z, U_{\delta(y)}(y) \cap bd S) .$$

Since, for  $\epsilon(y)$  small enough,  $\|u-z\| \geq \|\bar{u}-z\|$  where  $f(\bar{u}) = f^0$ ,  $\bar{u} = z + \alpha(u-z)$  and  $0 < \alpha \leq 1$ , we have  $\|u-z\| \geq \|w-z\|$ , i.e., (2.15). So, for all  $z \in U_{\epsilon(y)}(y) \cap \{z \mid \langle x-y, z-x \rangle \leq 0\}$ , (2.15) is true. Let  $\bar{y}$  be such a point that

$$x - \bar{y} \in (x - \hat{P}(x)) \cap L_{CG(x; \hat{P}(x))} .$$

Then, for any  $z \in U_{\epsilon(\bar{y})}(x) \cap \{z \mid \langle x-\bar{y}, z-x \rangle \leq 0\}$ , the relation

$$d(z, U_{\delta(\bar{y})}(\bar{y}) \cap bd S) \leq \varphi(x)^{1/2} \tag{2.16}$$

holds.

Since  $\dim L_{CG(x; \hat{P}(x))} \neq 0$ , there exists a finite set  $A \subset [G(x; \hat{P}(x)) \setminus \{x-\bar{y}\}] \cap L_{CG(x; \hat{P}(x))}$  such that  $\bar{y}-x \in CA$ , i.e., there exists a set of nonnegative scalars  $\{\lambda_1, \dots, \lambda_l\}$  satisfying  $\sum \lambda_i > 0$  and

$$\bar{y}-x = \sum \lambda_i \bar{u}_i, \quad \bar{u}_i \in A . \quad (2.17)$$

For any  $v \in \{z | \langle x-\bar{y}, z-x \rangle > 0\}$ , there exists at least one  $y(v) \in X-A$  such that

$$v \in \{z | \langle x-y(v), z-x \rangle < 0\} = H(x; x-y(v), \langle) . \quad (2.18)$$

In fact, from (2.17), one has the following relation

$$\langle x-\bar{y}, v-x \rangle > 0 \iff \langle \bar{y}-x, v-x \rangle = \sum_{\bar{u}_i \in A} \lambda_i \langle \bar{u}_i, v-x \rangle < 0 .$$

Therefore there exists at least one  $\bar{u}_i \in A$  such that

$$\langle \bar{u}_i, v-x \rangle < 0$$

This implies that there exists  $y(v) \in X-A$  such that (2.18) holds. Take a neighborhood

$$U_{\bar{\epsilon}}(x) \subset \left[ \bigcap_{y \in X-A} (U_{\epsilon(y)}(x)) \right] \cap (U_{\epsilon(\bar{y})}(x)) ,$$

where  $\bar{\epsilon}$  satisfies  $0 < \bar{\epsilon} < \min_{y \in (X-A) \cup \{\bar{y}\}} \epsilon(y)$  and  $\epsilon(y)$  satisfies (2.16). Of course, the following

relation is true

$$U_{\bar{\epsilon}}(x) \subset \bigcup_{y \in X-A} (U_{\epsilon(y)}(x)) \cup (U_{\epsilon(\bar{y})}(x)) .$$

It follows from this and (2.18) that there exists a neighborhood of  $x$ ,  $U_{\bar{\epsilon}}(x)$ , such that  $\forall v \in U_{\bar{\epsilon}}(x)$  one has  $\varphi(x)^{1/2} \geq \varphi(v)^{1/2}$ . So  $x$  is a locally optimal solution.  $\square$

### Optimality Condition 2.7

Suppose that  $x$  is a  $\wedge$ -point.  $x$  is a locally optimal solution to the problem (1.1) if and only if

$$\dim L_{CG(x; Y(x))} \neq 0 . \square$$

It is not easy to treat the situation in which  $G(x; \hat{P}(x)) \cap L_{CG(x; Y(x))} \neq \{\emptyset\}$  and  $CG(x; \hat{P}(x))$  is a pointed cone, but (2.12) is not valid. The main trouble is that in this case it is possible that although the main body is not able to expand in a straight line but it is able to expand in a curvilinear path. This situation will be discussed briefly. To this end we start with the following optimality condition (from [9]).

### Optimality Condition 2.8

A point  $x$  is a locally optimal solution if and only if there exists  $\epsilon > 0$  such that

$$\bigcap_{y \in Y(x)} [D_{\xi} + \|x - y\| B(0,1)]^c \cap (U_{\xi}(x)) = \emptyset . \square \quad (2.19)$$

Suppose  $y \in Y(x)$  and  $\zeta' \in bd [D_y + \|x - y\| B(0,1)] \cap S \cap U_{\delta(y)}(y)$ , where  $U_{\delta(y)}(y)$  is some neighborhood of  $y$  and  $D_{\xi}$  is the same as  $D_y$ .  $\zeta'$  can be expressed as

$$\zeta' = y' - \|x - y\| \nabla f(y') / \|\nabla f(y')\| \quad (2.20)$$

where  $y' \in \{y' | f(y') = f^0\}$ . For convenience subscripts of  $f$  and  $y$  are omitted for temporality. To begin with we approximate  $\nabla f(y') / \|\nabla f(y')\|$ . Since

$$\nabla f(y') = \nabla f(y) + H_y(y' - y) + \|y' - y\| \bar{o}(y, y' - y) . \quad (2.21)$$

where  $\bar{o}(y', y' - y) \in \mathbf{R}^n \rightarrow 0$ , as  $y' \rightarrow y$ .

Calculate the unit vector in (2.21)

$$\begin{aligned} \nabla f(y') / \|\nabla f(y')\| &= \nabla f(y) [1 - \langle \nabla f(y), H_y(y' - y) \rangle / \|\nabla f(y)\|^2 \\ &\quad + \|y' - y\| \bar{o}(y, y' - y)] / \|\nabla f(y)\| \\ &= \frac{\nabla f(y)}{\|\nabla f(y)\|} [1 - \langle \nabla f(y), H_y(y' - y) \rangle / \|\nabla f(y)\|^2] \\ &\quad + \frac{H_y(y' - y)}{\|\nabla f(y)\|} + \|y' - y\| \bar{o}(y, y' - y) . \end{aligned} \quad (2.22)$$

Substituting (2.22) into (2.20) we obtain the expression of  $\zeta'$

$$\begin{aligned} \zeta' &= y' - \frac{\|x - y\| \nabla f(y)}{\|\nabla f(y)\|} - \frac{\|x - y\| \nabla f(y) \langle \nabla f(y), H_y(y' - y) \rangle}{\|\nabla f(y)\|^3} \\ &\quad + \|x - y\| \frac{H_y(y' - y)}{\|\nabla f(y)\|} + \|y' - y\| \bar{o}(y, y' - y) \\ &= y' - (y - x) + q + \|y' - y\| \bar{o}(y, y' - y) . \end{aligned}$$

From this one has

$$\zeta' - x = y' - y + q + \|y' - y\| \bar{o}(y, y' - y) .$$

Let  $\zeta' - x = y' - y$ , then

$$\zeta \in f(\zeta - x + y) = f_y^0 = f(y) ,$$

and the following equations are satisfied

$$\zeta - x = \zeta - x + q + \|\zeta - x\| \bar{o}(y, \zeta - x)$$

In some neighborhood of  $x$  we make the approximation

$$\{\zeta | f(\zeta - x + y) = f(y)\} \cap U_\epsilon(x) = bd[D_y + \|x - y\| B(0,1)] \cap U_\epsilon(x)$$

Consequently, the set

$$\{\zeta | 2\nabla f(y)^T(\zeta - x) + (\zeta - x)^T H_y(\zeta - x) = 0\} \quad (2.23)$$

can be regarded as an approximation to  $bd[D_y + \|x - y\| B(0,1)] \cap S$  at  $x$ .

Finally combining (2.19) and (2.23), we obtain an approximate condition

$$\bigcap_{y_\xi \in Y(x)} \{z | 2\nabla f_\xi(y_\xi)(z - x) + (z - x)^T H_{y_\xi}(z - x) \geq 0\}^c \cap U_\epsilon(x) = \emptyset$$

i.e.

$$\bigcap_{y_\xi \in Y(x)} \{z | 2\nabla f_\xi(y_\xi)^T(z - x) + (z - x)^T H_{y_\xi}(z - x) < 0\} \cap U_\epsilon(x) = \emptyset . \quad (2.24)$$

Because  $\zeta - x = y - y$ , for some  $\epsilon > 0$  small enough, one has

$$\begin{aligned} & bd[D_\xi + \|x - y_\xi\| B(0,1)] \cap U_\epsilon(x) \\ & \subset \{z | 2\nabla f_\xi(y_\xi)^T(z - x) + (z - x)^T H_{y_\xi}(z - x) \leq 0\} \cap U_\epsilon(x) . \end{aligned}$$

Therefore

$$\begin{aligned} & \bigcap_{y_\xi \in Y(x)} [D_\xi + \|x - y_\xi\| B(0,1)]^c \cap U_\epsilon(x) \quad (2.25) \\ & \subset \bar{S} = \bigcap_{y_\xi \in Y(x)} \{z | 2\nabla f_\xi(y_\xi)^T(z - x) + (z - x)^T H_{y_\xi}(z - x) < 0\} \cap U_\epsilon(x) . \end{aligned}$$

It follows from (2.25) that if  $x$  is not a locally optimal solution to the problem (1.1) then the left hand side of (2.24) is nonempty. So if (2.24) holds then  $x$  must be an optimal solution (at least locally).

The problem now is how to verify if (2.24) holds when  $\dim N^+(x; Y(x)) \neq 0$  and  $n$ . Take appropriate  $\epsilon > 0$ , a point  $z_{N^+} \in N^+(x; Y(x)) \cap U_\epsilon(x)$  and find a  $\bar{z}$  such that it satisfies

$$d(z_{N^+}, \bar{z}) = d(z_{N^+}, \bar{S}) . \quad (2.26)$$

In order to avoid infinite programming brought by (2.25) and (2.26) the following sub-problem can be adopted, for some  $\xi \in \Omega(x)$

$$\min \|z - z_{N^+}\| \tag{2.27}$$

$$s.t. f_\xi(z - x + y_\xi) = f_\xi^0$$

$$f_\xi(z - x + y_\eta) \leq f_\eta^0 \quad \eta \in \Omega(x) \setminus \{\xi\}$$

with initial point  $x^{(0)} = x$ . A local solution  $z(x; \xi)$  for the problem (2.27) is near the boundary  $bd[D_\xi + \|x - y_\xi\| B(0,1)]$ . According to the information resulting from the point one can choose a method suitable for further checking if (2.24) holds. If needed, the parameter  $\epsilon$  used for controlling a neighborhood of  $x$  can be reduced successively.

When the region determined by convex constraints and complementary convex constraints and a norm associated with predefined unit convex body (in the case where it has a continuously differentiable surface corresponding to the unit convex body), the VD2 algorithm [1] described a method convergent to a locally optimal solution to the  $c_M$ -embedded problem (the DC problem). In this case the VD2 algorithm can be executed continuously without checking optimality conditions until meeting stopping criteria prescribed.

## 2. OTHER RESULTS

In this section some results related to checking optimality described in the last section will be presented. With this end in view we first give the following lemma.

**Lemma 3.1.** Suppose that  $y \in Q \subset \mathbb{R}^n$  and  $x \notin clQ$ . Then  $x - y \in L_{CG}(x; Q)$  if and only if both of  $Proj(x - y)/N^+(x; Q) = 0$  and  $Proj(y - x)/N^+(x; Q) = 0$  hold.

**Proof.** Since  $Proj(x - y)/N^+(x; Q) = 0$ , the hyperplane (or subspace)  $\{u \mid \langle u, x - y \rangle = 0\}$  separates  $x - y$  and  $N^+(x; Q)$  such that

$$\|x - y\|^2 > 0 \text{ and } \langle x - y, w \rangle \leq 0, \quad \forall w \in N^+(x; Q). \tag{3.1}$$

Likewise, for  $y - x$  it is the same as above, i.e.,

$$\|y - x\|^2 > 0 \text{ and } \langle y - x, w \rangle \leq 0, \quad \forall w \in N^+(x; Q). \tag{3.2}$$

From (3.1) and (3.2), we have

$$\langle x - y, w \rangle = 0, \quad \forall w \in N^+(x; Q).$$

Since  $\dim N^+(x; Q) + \dim L_{CG}(x; Q) = n$  and  $L_{CG}^\perp(x; Q) = LN^+(x; Q)$ , one has

$x-y \perp L_{CG(x;Q)}^\perp$ , i.e.,  $x-y \in L_{CG(x;Q)}$ . This is the proof of sufficiency. Conversely, it is easy to be proved.  $\square$

We now study with the aid of quadratic approximation how to analyze the local behavior of boundary at a point. According to this, an algorithm can be constructed, which can be used to identify the local behavior of boundary at a point.

Given a point  $y \in bd S$  and its corresponding constraint function  $f$ , and suppose further that

$$y'_T \in Tf(y) := \{y'_T | \langle y'_T - y, \nabla f(y) \rangle = 0\}$$

$$y' \in f(y') = f(y) ,$$

i.e.

$$y' \in \nabla f(y)^T (y' - y) + 1/2 (y' - y)^T H_y (y' - y) + \|y' - y\| o(y, y' - y) = 0 \quad (3.3)$$

$$y'_q \in q(y'_q; y) = f(y) ,$$

i.e.

$$y'_q \in \nabla f(y)^T (y'_q - y) + 1/2 (y'_q - y)^T H_y (y'_q - y) = 0. \quad (3.4)$$

where  $q(y; \cdot)$  is the approximation of order two for the Taylor expansion of  $f$  at  $y$ . For any  $y'_T \in Tf(y)$  in a neighborhood of  $y$ , we can find two corresponding points on  $f(y') = f(y)$  and  $q(y; y'_q) = f(y)$  respectively

$$y' = y'_T - \beta(y'_T) \nabla f(y) \quad (3.5)$$

and

$$y'_q = y'_T - \alpha(y'_T) \nabla f(y) . \quad (3.6)$$

$\alpha(y'_T)$  and  $\beta(y'_T)$  can be regarded as two mappings

$$\alpha: \mathbf{R}^{n-1} \rightarrow \mathbf{R} , \quad \beta: \mathbf{R}^{n-1} \rightarrow \mathbf{R} .$$

Substituting (3.5) into (3.3), one has

$$2 \nabla f(y)^T [y'_T - y - \beta(y'_T) \nabla f(y)] + (y'_T - y - \beta(y'_T) \nabla f(y))^T H_y (y'_T - y - \beta(y'_T) \nabla f(y)) + \|y' - y\| o(y, y' - y) = 0 .$$

From this we have

$$\begin{aligned} \beta(\mathbf{y}'_T)(\nabla f(\mathbf{y})^T H_{\mathbf{y}} \nabla f(\mathbf{y})) - 2\beta(\mathbf{y}'_T)(\|\nabla f(\mathbf{y})\|^2 + (\mathbf{y}'_T - \mathbf{y})^T H_{\mathbf{y}} \nabla f(\mathbf{y})) \\ + (\mathbf{y}'_T - \mathbf{y})^T H_{\mathbf{y}} (\mathbf{y}'_T - \mathbf{y}) + o(\|\mathbf{y}' - \mathbf{y}\|) = 0 . \end{aligned} \quad (3.7)$$

Suppose that  $\nabla f(\mathbf{y})$  is not self-conjugate to Hessian of  $f$  at  $\mathbf{y}$ . Solving (3.7), one has

$$\begin{aligned} \beta(\mathbf{y}'_T) = [(\|\nabla f(\mathbf{y})\|^2 + (\mathbf{y}'_T - \mathbf{y})^T H_{\mathbf{y}} \nabla f(\mathbf{y})) \pm ((\|\nabla f(\mathbf{y})\|^2 \\ + (\mathbf{y}'_T - \mathbf{y})^T H_{\mathbf{y}} \nabla f(\mathbf{y}))^2 - (\nabla f(\mathbf{y})^T H_{\mathbf{y}} \nabla f(\mathbf{y}))(\mathbf{y}'_T - \mathbf{y})^T H_{\mathbf{y}} (\mathbf{y}'_T - \mathbf{y}) \\ + o(\|\mathbf{y}' - \mathbf{y}\|^2))] / \nabla f(\mathbf{y})^T H_{\mathbf{y}} \nabla f(\mathbf{y}) . \end{aligned} \quad (3.8)$$

For the sake of simplicity, let

$$\begin{aligned} \mathbf{a} := (\mathbf{y}'_T - \mathbf{y}) / \|\nabla f(\mathbf{y})\| , \quad \mathbf{b} := \nabla f(\mathbf{y}) / \|\nabla f(\mathbf{y})\| , \\ \mathbf{c} := \mathbf{b}^T H_{\mathbf{y}} \mathbf{b} , \quad \mathbf{d} := \mathbf{a}^T H_{\mathbf{y}} \mathbf{a} . \end{aligned} \quad (3.9)$$

Made use of (3.9), (3.8) can be simplified as follows

$$\beta(\mathbf{y}'_T) = [1 + \mathbf{a}^T H_{\mathbf{y}} \mathbf{b} \pm ((1 + \mathbf{a}^T H_{\mathbf{y}} \mathbf{b})^2 - \mathbf{c} \mathbf{d} + o(\|\mathbf{y}' - \mathbf{y}\|^2))^{1/2}] / \mathbf{c} . \quad (3.10)$$

Making Taylor approximation of second order, one has

$$\begin{aligned} \beta(\mathbf{y}'_T) = \frac{1}{\mathbf{c}} (1 + \mathbf{a}^T H_{\mathbf{y}} \mathbf{b}) \\ - \left[ 1 + \mathbf{a}^T H_{\mathbf{y}} \mathbf{b} + \frac{1}{2} \left[ \mathbf{a}^T H_{\mathbf{y}} \mathbf{b} \mathbf{b}^T H_{\mathbf{y}} \mathbf{a} - \mathbf{c} \mathbf{d} \right] - \frac{1}{8} \left[ 2 \mathbf{a}^T H_{\mathbf{y}} \mathbf{b} + \dots \right]^2 + o(\|\mathbf{y}' - \mathbf{y}\|^2) \right] / \mathbf{c} . \end{aligned} \quad (3.11)$$

Because  $(\mathbf{y}'_T - \mathbf{y}) / (\mathbf{y}' - \mathbf{y}) \rightarrow 1$  as  $\mathbf{y}'_T \rightarrow \mathbf{y}$ , the "-" is taken in (3.10). Therefore,

$$\begin{aligned} \beta(\mathbf{y}'_T) = \frac{1}{2} \mathbf{d} + o(\|\mathbf{y}'_T - \mathbf{y}\|^2) \\ = \frac{1}{2} (\mathbf{y}'_T - \mathbf{y})^T H_{\mathbf{y}} (\mathbf{y}'_T - \mathbf{y}) / \|\nabla f(\mathbf{y})\|^2 + o_{\beta}(\|\mathbf{y}'_T - \mathbf{y}\|^2) . \end{aligned} \quad (3.12)$$

For (3.4) and (3.6), we have the similar expansion

$$\alpha(\mathbf{y}'_T) = \frac{1}{2} (\mathbf{y}'_T - \mathbf{y})^T H_{\mathbf{y}} (\mathbf{y}'_T - \mathbf{y}) / \|\nabla f(\mathbf{y})\|^2 . \quad (3.13)$$

Define

$$\Delta(\mathbf{y}'_T) := \frac{1}{2} (\mathbf{y}'_T - \mathbf{y})^T H_{\mathbf{y}} (\mathbf{y}'_T - \mathbf{y}) / \|\nabla f(\mathbf{y})\|^2 \quad (3.14)$$

$$\mathbf{y}'_T \in T f(\mathbf{y}) \cap U_{\epsilon}(\mathbf{y}) , \text{ for some } \epsilon > 0 .$$

The foregoing development is summarized in the following lemma.

**Lemma 3.2.** Suppose  $y \in Y(x)$ . If  $y \in \check{P}(x) (\hat{P}(x) \text{ or } \bar{P}(x))$ , then there exists a  $U_\epsilon(y)$  such that  $\Delta(y_T') \leq 0$  ( $\geq 0$ ) for all  $y_T' \in Tf(y) \cap U_\epsilon(y)$ . Conversely, if there exists a  $U_\epsilon(y)$  such that, for all  $y_T' \in Tf(y) \cap U_\epsilon(y)$ ,  $\Delta(y_T') > 0$  ( $< 0$ ), then  $y \in \hat{P}(x)$  or  $\bar{P}(x)$  ( $\check{P}(x)$ ).  $\square$

According to our hypothesis that for any boundary point  $y \in bd S$ ,  $\nabla f(y) \neq 0$ , for any  $y \in bd S$  there exists a matrix

$$C(y) = [u_1, \dots, u_{n-1}] ,$$

that is, the matrix consisting of vectors in a basis of subspace  $Tf(y) - y$ , such that, for each  $y_T' \in Tf(y)$ , there is the unique expression

$$y_T' - y = \|\nabla f(y)\| Cw , \quad w \in Tf(y) - y = \mathbf{R}^{n-1} . \quad (3.15)$$

Substituting (3.15) into (3.14), we get

$$\Delta = w^T C^T H_y C w . \quad (3.16)$$

From Lemma 3.2 and (3.16), it is easy to prove the following theorem.

**Theorem 3.3.** Suppose  $y \in Y(x)$ . If  $y \in \check{P}(x) (\hat{P}(x) \text{ or } \bar{P}(x))$ , then  $C^T H_y C \leq 0$  ( $\geq 0$ ). Conversely, if  $C^T H_y C < 0$  ( $> 0$ ), then  $y \in \check{P}(x) (\hat{P}(x) \text{ or } \bar{P}(x))$ . If  $C^T H_y C$  is indefinite, then  $y \in \tilde{P}(x)$ .  $\square$

Note that although from  $C^T H_y C \geq 0$ , we are not able to conjecture if  $y$  must belong to  $\hat{P}(x)$ . For convenience,  $y$  is put into  $\hat{P}(x)$  because of  $\beta(y_T') = o(\|y_T' - y\|^2)$  for every  $y_T' \in Tf(y) \cap U_\epsilon(y)$  when  $\epsilon$  is small enough.

Now we will discuss an important problem, but a very difficult one as well. How to find  $Y(x)$ ? Clearly, it is the problem we are interested in. Suppose that the set  $\Omega(x)$  has been found. For every  $\xi \in \Omega(x)$ , consider the subproblem

$$\begin{aligned} \min \|y - x\|^2 & \quad (3.17) \\ \text{s.t. } f_\xi(y) &= f_\xi^o , \\ f_\eta(y) &\leq f_\eta^o , \quad \eta \in \Omega(x) \setminus \{\xi\} . \end{aligned}$$

The problem in question can be described as finding

$$Y(x) = \bigcup_{\xi \in \Omega(x)} Y(x, \xi)$$

$$= \bigcup_{\xi \in \Omega(x)} \{ \bar{y} \in \mathbf{R}^n \mid \bar{y} \text{ is a solution to the problem (3.17)} \} .$$

The part of inequality constraints in (3.17) can be taken no account of, or straightforwardly, they can be dropped when finding  $Y(x, \xi)$ , i.e., the subproblem

$$\begin{aligned} \min \|y - x\|^2 \\ \text{s.t. } f_\xi(y) = f_\xi^0 \end{aligned} \quad (3.18)$$

will be considered. It is evident that (3.17) and (3.18) are equivalent. We assume that  $Y(x, f)$  is finite. Then

$$Y(x) = \hat{P}(x) \bigcup \bigvee P(x) \bigcup \tilde{P}(x) .$$

For convenience, the subscript in (3.18) is omitted and  $Y(x, f)$  is used instead of  $Y(x, \xi)$  temporarily. According to the definitions of  $\hat{P}(x)$ ,  $\bigvee P(x)$  and  $\tilde{P}(x)$  for each  $\bar{y} \in Y(x, f)$ , there exists a  $\delta'_0(\bar{y}) > 0$  such that

$$\text{bd } S \cap (U_{\delta'_0(\bar{y})}(\bar{y})) \cap \text{bd } B(x, \varphi(x)^{1/2}) = \{ \bar{y} \} .$$

In other words, there exists a  $\delta_0(\bar{y}) > 0$  such that

$$Y(x, f) \cap H^+(\bar{y} - \delta_0(\bar{y}) \nabla f(\bar{y}) ; \nabla f(\bar{y})) = \{ \bar{y} \} ,$$

where

$$H^+(\bar{y} - \delta_0(\bar{y}) \nabla f(\bar{y}) ; \nabla f(\bar{y})) = \{ z \in \mathbf{R}^n \mid \langle z - \bar{y} + \delta_0(\bar{y}) \nabla f(\bar{y}) , \nabla f(\bar{y}) \rangle \geq 0 \} .$$

There exists a set  $\{ \delta_0(\bar{y}) \mid \bar{y} \in Y(x, f) \}$  such that

$$\bigcap_{\bar{y} \in Y(x, f)} H^+(\bar{y} - \delta_0(\bar{y}) \nabla f(\bar{y}) ; -\nabla f(\bar{y})) \cap B(x, \varphi(x)^{1/2}) \cap Y(x, f) = \emptyset .$$

Define

$$X = \bigcap_{\bar{y} \in Y(x, f)} H^+(\bar{y} - \delta_0(\bar{y}) \nabla f(\bar{y}) ; -\nabla f(\bar{y})) \cap B(x, \varphi(x)^{1/2}) . \quad (3.19)$$

It is closed, convex and bounded. Clearly

$$d(X, f(y) = f^0) > 0 .$$

It can be proved that, if  $Y(x, f)$  is finite, then there exists a polyhedron  $P_l$  such that

$$X \subset \text{int} P_l \text{ and } f(y) = f^0 \subset P_l^c ,$$

i.e.,  $bd P_l$  can separate  $X$  and  $f(y) = f^0$  strictly.

Given  $Q \neq \emptyset$  and  $Q \subset Y(x, f)$ , consider the following problem

$$\begin{aligned} & \min \|y - x\|^2 \\ & \text{s.t. } f(y) = f^0, \\ & y \in \bigcap_{\bar{y} \in Q} H^+(\bar{y} - \delta_0(\bar{y}) \nabla f(\bar{y}); -\nabla f(\bar{y})) \quad (SC) \quad (3.20) \\ & (y \in M \text{ if needed}) \end{aligned}$$

where  $M$  is a bounded domain containing  $B(x, \varphi(x)^{1/2})$ .

Suppose that  $Y(x, f)$  is finite and  $Y(x, f) \setminus Q \neq \emptyset$ . The set determined by the constraints of (3.20) is nonempty, i.e.,

$$\{y \mid f(y) = f^0 \cap (\bigcap_{\bar{y} \in Q} H^+(\bar{y} - \delta_0(\bar{y}) \nabla f(\bar{y}); -\nabla f(\bar{y})))\} \neq \emptyset. \quad (3.21)$$

After solved (3.20) each time two cases may happen. One is that  $y^* \in bd B(x, \varphi(x)^{1/2})$ , the other is that  $y^* \in bd B(x, \varphi(x)^{1/2})$ , i.e.,  $d(y^*, x) > \varphi(x)^{1/2}$  where  $y^*$  is a local solution to (3.20), obtained after some algorithms were executed for solving (3.20), e.g., [ 12 ].

**Case I.** In this case, a new touching point has been obtained. The set  $Q$  can be enlarged after  $Q = Q \cup \{y^*\}$ . If there is a need, a new hyperplane

$$Tf(y^*) - \delta_0(y^*) \nabla f(y^*)$$

can be introduced and

$$H^+(y^* - \delta_0(y^*) \nabla f(y^*); -\nabla f(y^*))$$

can be added to the original constraint set of the problem (3.20). Resolve the problem again.

**Case II.**  $y^* \in bd B(x, \varphi(x)^{1/2})$ . A hyperplane

$$\Pi(y^*) = \{z \mid \langle z - y^*, x - y^* \rangle - \delta_0(y^*) \|x - y^*\|^2 = 0\}$$

can be constructed where  $\delta_0(y^*)$  satisfies

$$\delta(y^*) \|x - y^*\| \geq (1/2) [\|x - y^*\| - \varphi(x)^{1/2}] \quad (3.22)$$

and

$$B(x, \varphi(x)^{1/2}) \subset H^+(y^* + \delta_0(y^*)(x - y^*); x - y^*).$$

Then the right-hand side of the above inclusion relation is added to the set of original constraints of the problem (3.20). Resolve the problem (3.20) with the enlarged constraint set and an appropriately initial point  $y^*(0)$  such that

$$\|x - y_{(0)}^*\| \leq (1/2)[\|x - y^*\| + \varphi(x)^{1/2}] . \quad (3.23)$$

From this, if (3.20) is resolved infinitely, then a sequence of solutions can be formed, denoted by  $\{y_k^*\}_1^\infty$ . In terms of (3.23), one has

$$\begin{aligned} 0 \leq \|y_k^* - x\| - \varphi(x)^{1/2} &\leq 1/2[\|x - y_{k-1}^*\| - \varphi(x)^{1/2}] \\ &\leq (1/2)^{k-1}[\|x - y_1^*\| - \varphi(x)^{1/2}] . \end{aligned}$$

So  $\{y_k^*\} \rightarrow y^{**} \in B(x, \varphi(x)^{1/2})$ . Summarizing the description above, we get the following lemma.

**Lemma 3.5.** If  $Y(x, f)$  is finite and  $Y(x, f) \setminus Q \neq \emptyset$  and  $Q \subset Y(x, f)$ , then, by solving (3.20) repeatedly, a new touching point can be found provided that initial points in an infinitely iterative process satisfy (3.23).  $\square$

There are two points we have to mention before a strategy is proposed. The first is that, if  $\bar{y} \in Q$  and for some  $\delta > 0$ , the intersection of the hyperplane  $Tf(\bar{y}) - \delta \nabla f(\bar{y})$  and the hypersurface is empty, i.e.

$$(Tf(\bar{y}) - \delta \nabla f(\bar{y})) \cap (f(y) = f^0) = \emptyset .$$

then the hypersurface is included in the halfspace  $H^+(\bar{y} - \delta \nabla f(\bar{y}); \nabla f(\bar{y}))$ , because of  $f \in C^2$ . The second point concerns a basis in  $(n-1)$ -dimensional space. Suppose that the columns of  $C(\bar{y}) = [u_1, \dots, u_{n-1}]$  form a basis in  $Tf(\bar{y}) - \bar{y}$ . For any  $\delta > 0$ ,  $\{u_1, \dots, u_{n-1}\}$   $C(\bar{y}) + \bar{y} - \delta \nabla f(\bar{y})$  is a basis on the hyperplane (manifold)  $Tf(\bar{y}) - \delta \nabla f(\bar{y})$ . In addition to these, an engineering infinite  $\infty_E$  is defined by a positive number large enough,  $\tau_\infty$ , i.e.,  $\infty_E := \tau_\infty$  controlled by  $M$  in (3.20).

Finally it is necessary to mention the case in which  $Y(x, f)$  is an infinite set. But it is not our purpose to elaborate on the details. We only make a short discussion below.

It is clear that

$$Y(x, f) \subset A := \left[ \bigcup_{y \in Y(x, f)} (U_{\epsilon(y)}(y)) \right] \cap \text{bd } B(x, \varphi(x)^{1/2}) , \quad \epsilon(y) > 0$$

and

$$N^+(x; Y(x, f)) \supset N^+(x; A) .$$

Since  $bd B(x, \varphi(x)^{1/2})$  and  $Y(x, f)$  are compact, there exists a finite family of neighborhoods

$$\{U_{\epsilon(y)}(y) | y \in Q \subset Y(x, f), \epsilon(y) > 0\} .$$

covering  $Y(x, f)$ , where  $Q$  is a finite subset of  $Y(x, f)$ . Let  $\epsilon = \max\{\epsilon(y) | y \in Q\}$ . Then

$$Y(x, f) \subset \bigcup_{y \in Q} (U_{\epsilon}(y)) , \quad Q \subset Y(x, f) .$$

In general, we are not able to infer from  $Q \subset Y(x, f)$  that

$$N^+(x; Y(x, f)) \supset N^+(x; \bigcup_{y \in Q} U_{\epsilon}(y))$$

for some  $\epsilon > 0$ . A finite subset of  $Y(x, f)$ ,  $Q$ , is called a finite  $N^+ - \epsilon$  approximation to  $Y(x, f)$  if

$$N^+(x; Y(x, f)) \supset N^+(x; \bigcup_{y \in Q} U_{\epsilon}(y)) . \quad (3.25)$$

**Theorem 3.8.** Suppose that  $Q_i \subset Y(x, f)$ ,  $i = 0, 1, \dots, \{Q_i\}_0^{\infty} \uparrow$  in the sense of the inclusion relation,  $\{\epsilon_i\}_0^{\infty} \downarrow 0$  and  $Y(x, f)$  is infinite. If for any  $i = 0, 1, \dots$ , the following condition is satisfied

$$Y(x, f) \subset \bigcup_{y \in Q_i} (U_{\epsilon_i}(y)) , \quad (3.26)$$

then

$$\lim_{i \rightarrow \infty} N^+(x; Q_i) = N^+(x; Y(x, f)) . \quad (3.27)$$

**Proof.** From (3.26), one has

$$Y(x, f) \subset \bigcap_{i=0}^{\infty} \bigcup_{y \in Q_i} (U_{\epsilon_i}(y)) . \quad (3.28)$$

Now we prove the inclusion relation opposite to (3.28). For any fixed  $z$  such that

$$z \in \bigcap_{i=0}^{\infty} \bigcup_{y \in Q_i} (U_{\epsilon_i}(y))$$

and any  $i$ , there exist  $y_i \in Q_i$  and  $\alpha_i$  such that  $0 \leq \alpha_i \leq \epsilon_i$  and

$$z = y_i + \alpha_i \omega_i ,$$

where  $\|\omega_i\| \leq 1$ . From this we have

$$\|z - y_i\| < \epsilon_i .$$

Since  $\epsilon_i \rightarrow 0$  and  $Y(x, f)$  is compact,  $\{y_i\}_0^\infty \rightarrow z$  and  $z \in Y(x, f)$ . Hence

$$Y(x, f) = \bigcap_{i=0}^{\infty} \bigcup_{y \in Q_i} (U_{\epsilon_i}(y)) . \quad (3.29)$$

It can be seen that  $\lim_{i \rightarrow \infty} Q_i$  is dense in  $Y(x, f)$  because of  $\{Q_i\} \uparrow$  and (3.29). Thus

$$N^+(x; \lim_{i \rightarrow \infty} Q_i) = N^+(x; Y(x, f)) \quad (3.30)$$

because of  $cl \lim_{i \rightarrow \infty} Q_i = Y(x, f)$ . Note that  $\{N^+(x; Q_i)\} \downarrow$  as  $\{Q_i\} \uparrow$ . It is easy to prove that

$$N^+(x; \lim_{i \rightarrow \infty} Q_i) = N^+(x; \bigcup_{i=0}^{\infty} Q_i) = \bigcap_{i=0}^{\infty} N^+(x; Q_i) = \lim_{i \rightarrow \infty} N^+(x; Q_i) . \quad (3.31)$$

In view of (3.30) and (3.31), one has (3.27). The proof is completed.  $\square$

#### Acknowledgments.

We are grateful to Prof. Hiriart-Urruty, Dr. S. Uryas'ev and Mr. L. Wu for their many helpful suggestions and comments.

#### References

- [1] L.M. Vidigal and S.W. Director, "A design centering algorithm for non-convex regions of acceptability", IEEE Trans. on Computer-Aided Design of Integrated Circuits and Systems CAD-1 (1982) 13-24.
- [2] L.M. Vidigal and S.W. Director, "Comparison of two methods for design centering", IEEE CH1635-2 (1981) 843-846.
- [3] R.K. Brayton, S.W. Director, G.D. Hachtel and L.M. Director, "A new algorithm for statistical circuit design based on quasi-Newton methods and function splitting", IEEE Trans. Circuits Syst. CAS-26 (1979) 784-794.
- [4] R.K. Brayton, S.W. Director and G.D. Hachtel, "Yield maximization and worst-case design with arbitrary statistical distributions", IEEE Trans. Circuits Syst. CAS-27 (1980) 756-764.
- [5] Elijah Polak and Alberto Sangiovanni-Vincentelli, "Theoretical and computational aspects of the optimal design centering, tolerancing and turning problem", IEEE Trans. Circuits Syst. CAS-26 (1979) 795-813.
- [6] V.F. Demyanov and V.N. Malozemov, Introduction to Minimax (Wiley, New York, 1974).
- [7] B.N. Pshenichnyi, Necessary conditions for extremum problems (Marcel Dekker, New York, 1971).
- [8] Z.Q. Xia, J.-J. Strodiot and V.H. Nguyen, "On a proof of the sufficient condition mentioned in the VD2 algorithm", to appear.

- [9] Z.Q. Xia, J.-J. Strodiot and V.H. Nguyen, "On expansible directions for the norm bodies in the DC problem", to appear.
- [10] Z.Q. Xia, J.-J. Strodiot and V. Hien Nguyen, "The  $C_M$ -embedded problem and a sufficient optimality condition", Report, International Institute for Applied Systems Analysis, Laxenburg, Austria, (1986).
- [11] J. Stoer and C. Witzgall, Convexity and Optimization (Springer-Verlag, New York, 1970).
- [12] M.J.D. Powell, "Algorithm for nonlinear optimization that use Lagrangian functions", Math. Programming 14 (1978) 224-248.
- [13] V.F. Demyanov and L.V. Vasiliev, Nondifferentiable Optimization (Nauka, Moscow, 1981, and English version by T. Sasagawa, Optimization Software, Inc. Publications Division, New York, 1985.)
- [14] H. Tuy, "A general deterministic to global optimization via D.C. programming" Report, Institute of Mathematics, Hanoi.