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**Rockafellar, R.T. & Wets, R.J.-B.**

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# ***WORKING PAPER***

## **GENERALIZED LINEAR-QUADRATIC PROBLEMS OF DETERMINISTIC AND STOCHASTIC OPTIMAL CONTROL IN DISCRETE TIME**

*R.T. Rockafellar  
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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS  
A-2361 Laxenburg, Austria

## **FOREWORD**

The study and control of dynamical systems is an important part of the program of the Systems and Decision Sciences project at IIASA. In this report the authors are concerned with the properties of a class of linear-quadratic dynamical systems that are subject to random disturbances. Optimality conditions are derived in a form that emphasizes the possibilities of decomposition, a major step in the development of solution procedures for such classes of problems.

Alexander B. Kurzhanski  
Chairman  
System and Decision Sciences Program

# Generalized Linear-Quadratic Problems of Deterministic and Stochastic Optimal Control in Discrete Time

*R.T. Rockafellar\* and R.J-B. Wets\**

**Abstract.** Two fundamental classes of problems in large-scale linear and quadratic programming are described. Multistage problems covering a wide variety of models in dynamic programming and stochastic programming are represented in a new way. Strong properties of duality are revealed which support the development of iterative approximate techniques of solution in terms of saddlepoints. Optimality conditions are derived in a form that emphasizes the possibilities of decomposition.

**Keywords:** discrete-time optimal control, dynamic programming, stochastic programming, large-scale linear-quadratic programming, intertemporal optimization, finite generation method.

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## 1. Introduction

The importance of linear and quadratic programming problems is well appreciated in finite-dimensional optimization. Such problems serve as mathematical models in their own right and as subproblems solved within the context of general numerical methods of nonlinear programming. In optimal control only a relatively small class of linear-quadratic problems has traditionally received much attention, however. A much more general class has recently been explored by Rockafellar [1] with the aim of opening up a wide domain for application of techniques of large-scale linear and quadratic programming, in particular the finite generation method of Rockafellar and Wets [2], [3], [4] that has been implemented in stochastic programming [5]. Central to this purpose is the development of flexible problem formulations for which there is a strong duality theory that represents optimal trajectories and controls in terms of saddlepoints of a “decomposable” Lagrangian.

In the present paper a discrete-time version of the deterministic models in [1] is investigated and corresponding results on optimality and duality are obtained. The formulations and results are then generalized to the stochastic case. The focus on discrete time is motivated by the computational possibilities already mentioned, so we do not hesitate to suppose also that the probability space for our stochastic version is discrete.

Our emphasis is on setting up a general framework for large-scale finite-dimensional linear-quadratic programming problems that reflect the special structure of optimal control. Besides being useful for numerical experimentation, such a framework may stimulate new applications, for instance in areas like operations research and resource systems management, where inequality constraints occur that jointly involve states and controls. Although the task of clarifying the relationship between finite and infinite-dimensional formulations is an important one, it is not the object of our efforts here.

In fact our discrete-time problems are more general than typical continuous-time problems in one respect: the dimensionality of the state and control vectors can vary with time. This feature is important in multistage modeling, where the decision structure in one period need not be the same as in another. The flexibility it provides allows us to show that a much wider class of problems is covered by our format than might at first be imagined.

## 2. Generalized Linear-Quadratic Programming.

The control problems that will be formulated are based on a concept of generalized linear-quadratic programming explained fully in Rockafellar [1]. A problem fits this concept if it can be expressed in the form

$$(\mathcal{P}) \quad \text{minimize } f(u) = \sup_{v \in V} J(u, v) \text{ over all } u \in U,$$

where  $U$  and  $V$  are polyhedral convex sets in  $\mathbf{R}^k$  and  $\mathbf{R}^\ell$ , and  $J$  is a quadratic convex-concave function on  $U \times V$ , namely

$$(2.1) \quad J(u, v) = p \cdot u + \frac{1}{2}u \cdot Pu + q \cdot v - \frac{1}{2}v \cdot Qv - v \cdot Du,$$

where  $P$  and  $Q$  are symmetric and positive *semidefinite* (possibly 0—we do not exclude “linear” when we say “quadratic”, as we try to underline by sometimes using the term “linear-quadratic”). The problem dual to  $(\mathcal{P})$  is then

$$(\mathcal{Q}) \quad \text{maximize } g(v) = \inf_{u \in U} J(u, v) \text{ over all } v \in V.$$

Here  $f(u)$  could be  $\infty$  and  $g(v)$  could be  $-\infty$ . We regard  $u$  as a feasible solution to  $(\mathcal{P})$  only if  $u \in U$  and  $f(u) < \infty$ ; likewise, we regard  $v$  as a feasible solution to  $(\mathcal{Q})$  only if  $v \in V$  and  $g(v) > -\infty$ .

The expression of problems  $(\mathcal{P})$  and  $(\mathcal{Q})$  is facilitated by the notation

$$(2.2) \quad \rho_{V,Q}(r) = \sup_{v \in V} \{r \cdot v - \frac{1}{2}v \cdot Qv\} \text{ for } r \in \mathbf{R}^\ell,$$

$$(2.3) \quad \rho_{U,P}(s) = \sup_{u \in U} \{s \cdot u - \frac{1}{2}u \cdot Pu\} \text{ for } s \in \mathbf{R}^k.$$

Thus  $\rho_{V,Q}$  is a function on  $\mathbf{R}^\ell$  determined by the specification of a polyhedral convex set  $V \subset \mathbf{R}^\ell$  and a symmetric positive semidefinite matrix  $Q \in \mathbf{R}^{\ell \times \ell}$ . It is in general “piecewise linear-quadratic” in a sense made precise in [1], and it may take on the value  $\infty$ . There are many special cases deserving of mention, but for these too one should consult to [1]. Let it suffice to observe that when  $0 \in V$ , one has  $\rho_{V,Q}(r) \geq 0$  for all  $r$ ,  $\rho_{V,Q}(0) = 0$ . Then  $\rho_{V,Q}(r)$  can be interpreted as an expression that “monitors deviations of  $r$  from 0”. Similarly for  $\rho_{U,P}$ .

In this notation our general problems can be written as

$$(\mathcal{P}) \quad \text{minimize } p \cdot u + \frac{1}{2}u \cdot Pu + \rho_{V,Q}(q - Du) \text{ over } u \in U,$$



$$(Q) \quad \text{maximize } q \cdot v - \frac{1}{2} v \cdot Q v - \rho_{U,P}(D^* v - p) \text{ over } v \in V$$

(where the asterisk \* signals the transpose matrix). In (P), therefore, one has the possibility of linear constraints represented by the condition  $u \in U$ , and also an objective term which “monitors deviations of  $Du$  from  $q$ ”. This may be a penalty term that is zero for some kinds of deviations but positive for others. For example, if  $V = \mathbb{R}_+^\ell$ ,  $Q = 0$ , one has

$$(2.4) \quad \rho_{V,Q}(q - Du) = \begin{cases} 0 & \text{if } Du \geq q, \\ \infty & \text{if } Du \not\geq q, \end{cases}$$

so that the  $\rho$  term in (P) is a “sharp” representation of the constraint  $Du \geq q$ . If at the same time one has  $U = \mathbb{R}_+^k$ ,  $P = 0$ , then similarly

$$(2.5) \quad \rho_{U,P}(D^* v - p) = \begin{cases} 0 & \text{if } D^* v \leq p, \\ \infty & \text{if } D^* v \not\leq p. \end{cases}$$

In this case (P) and (Q) reduce to a canonical pair of linear programming problems in duality. See [1] for discussion of the rich possibilities that such  $\rho$  terms provide more generally in mathematical modeling.

The basic facts about the relationship between (P) and (Q) can be derived from the standard theory of linear and quadratic programming, specifically the duality theorem of Cottle [6] and the existence theorem of Frank and Wolfe [7].

**Theorem 2.1** (Rockafellar and Wets [3, Theorem 2]). *If either (P) or (Q) has finite optimal value, or if both problems have feasible solutions, then both optimal values are finite and equal, and both problems have optimal solutions. In this case a pair  $(\bar{u}, \bar{v})$  is a saddlepoint of  $J(u, v)$  relative to  $u \in U$  and  $v \in V$  if and only if  $\bar{u}$  is an optimal solution to (P) and  $\bar{v}$  is an optimal solution to (Q).*

### 3. Deterministic Control Model.

We want now to formulate problems in this vein that belong to optimal control. The dynamical system we consider takes the form

$$(3.1) \quad \begin{aligned} x_\tau &= A_\tau x_{\tau-1} + B_\tau u_\tau + b_\tau \text{ for } \tau = 1, \dots, T, \\ x_0 &= B_0 u_0 + b_0, \quad \text{where } u_\tau \in U_\tau \text{ for } \tau = 0, 1, \dots, T. \end{aligned}$$

The vectors  $u_\tau \in \mathbb{R}^{k_\tau}$  are controls, and the vectors  $x_\tau \in \mathbb{R}^{n_\tau}$  are states (observe that dimensions can vary with  $\tau$ ). We write  $u = (u_0, u_1, \dots, u_T)$  and  $x = (x_0, x_1, \dots, x_T)$ . Thus  $x$  is uniquely determined by  $u$ , and the transformation  $u \mapsto x$  is affine. Note that  $u_0$  serves as a supplementary parameter vector more than as a control vector in the usual dynamical sense.

The sets  $U_\tau \subset \mathbb{R}^{k_\tau}$  are assumed to be polyhedral convex (nonempty). The matrices  $A_\tau$ ,  $B_\tau$  and vectors  $b_\tau$  are of appropriate dimension:

$$A_\tau \in \mathbb{R}^{n_\tau \times n_{\tau-1}}, \quad B_\tau \in \mathbb{R}^{n_\tau \times k_\tau}, \quad b_\tau \in \mathbb{R}^{n_\tau}.$$

(By taking  $k_0 = 0$ , one could eliminate  $u_0$  from (3.1) and have  $x_0 = b_0$ .)

Our deterministic control problem is:

$$(P_{\text{det}}) \quad \begin{aligned} &\text{minimize subject to (3.1) the expression } f(u) = \\ &\sum_{\tau=0}^T [p_\tau \cdot u_\tau + \frac{1}{2} u_\tau \cdot P_\tau u_\tau - c_{\tau+1} \cdot x_\tau] \\ &+ \sum_{\tau=1}^T \rho_{V_\tau, Q_\tau} (q_\tau - C_\tau x_{\tau-1} - D_\tau u_\tau) + \rho_{V_{T+1}, Q_{T+1}} (q_{T+1} - C_{T+1} x_T). \end{aligned}$$

Here  $V_\tau$  is a polyhedral convex set (nonempty) in  $\mathbb{R}^{\ell_\tau}$ , and the matrices  $P_\tau$  and  $Q_\tau$  are symmetric and positive semidefinite. One has

$$P_\tau \in \mathbb{R}^{k_\tau \times k_\tau}, \quad Q_\tau \in \mathbb{R}^{\ell_\tau \times \ell_\tau}, \quad p_\tau \in \mathbb{R}^{k_\tau}, \quad q_\tau \in \mathbb{R}^{\ell_\tau},$$

$$c_\tau \in \mathbb{R}^{n_{\tau-1}}, \quad C_\tau \in \mathbb{R}^{\ell_\tau \times n_{\tau-1}}, \quad D_\tau \in \mathbb{R}^{\ell_\tau \times k_\tau}.$$

In this notation the elements  $A_\tau$  and  $D_\tau$  are defined only for  $\tau = 1, \dots, T$ , but  $B_\tau, b_\tau, P_\tau, p_\tau$ , are defined for  $\tau = 0, 1, \dots, T$  and  $C_\tau, c_\tau, Q_\tau, q_\tau$  for  $\tau = 1, \dots, T, T+1$ .

For the problem that will turn out to be dual to  $(P_{\text{det}})$ , the dynamical system goes backward in time:

$$(3.2) \quad \begin{aligned} y_\tau &= A_\tau^* y_{\tau+1} + C_\tau^* v_\tau + c_\tau \text{ for } \tau = 1, \dots, T, \\ y_{T+1} &= C_{T+1}^* v_{T+1} + c_{T+1}, \quad \text{where } v_\tau \in V_\tau \text{ for } \tau = 1, \dots, T, T+1. \end{aligned}$$

The vectors  $v_\tau \in \mathbb{R}^{\ell_\tau}$  are the dual controls, and the vectors  $y_\tau \in \mathbb{R}^{n_{\tau-1}}$  are the dual states. We write

$$v = (v_1, \dots, v_T, v_{T+1}) \text{ and } y = (y_1, \dots, y_T, y_{T+1}).$$

The dual problem then is

$$\begin{aligned} & \text{maximize subject to (3.2) the expression } g(v) = \\ (Q_{\det}) \quad & \sum_{\tau=1}^{T+1} [q_\tau \cdot v_\tau - \tfrac{1}{2} v_\tau \cdot Q_\tau v_\tau - b_{\tau-1} \cdot y_\tau] \\ & - \sum_{\tau=1}^T \rho_{U_\tau, P_\tau} (B_\tau^* y_{\tau+1} - D_\tau^* v_\tau - p_\tau) - \rho_{U_0, P_0} (B_0^* y_1 - p_0). \end{aligned}$$

In this formula  $y$  is the trajectory uniquely determined from  $v$  by (3.2).

**Proposition 3.1.** Suppose  $x$  corresponds to  $u$  by (3.1), and  $y$  to  $v$  by (3.2). Then

$$(3.3) \quad \sum_{\tau=0}^T y_{\tau+1} \cdot [B_\tau u_\tau + b_\tau] = \sum_{\tau=1}^{T+1} x_{\tau-1} \cdot [C_\tau^* v_\tau + c_\tau].$$

**Proof.** In view of the relations (3.1) the left side of (3.3) can be written as

$$\begin{aligned} & y_1 \cdot x_0 + \sum_{\tau=1}^T y_{\tau+1} [x_\tau - A_\tau x_{\tau-1}] \\ & = y_1 \cdot x_0 + y_2 \cdot x_1 + \dots + y_{T+1} \cdot x_T - \sum_{\tau=1}^T x_{\tau-1} \cdot A_\tau^* y_{\tau+1}. \end{aligned}$$

Likewise from (3.2) the right side becomes

$$\begin{aligned} & x_T \cdot y_{T+1} + \sum_{\tau=1}^T x_{\tau-1} \cdot [y_\tau - A_\tau^* y_{\tau+1}] \\ & = y_1 \cdot x_0 + y_2 \cdot x_1 + \dots + y_{T+1} \cdot x_T - \sum_{\tau=1}^T x_{\tau-1} \cdot A_\tau^* y_{\tau+1}. \end{aligned}$$

Thus the two sides are equal, as claimed.  $\square$

**Proposition 3.2.** Let  $U = U_0 \times \dots \times U_T$  and  $V = V_1 \times \dots \times V_{T+1}$ , and for  $u \in U$  and  $v \in V$  define

$$\begin{aligned} (3.4) \quad J(u, v) = & \sum_{\tau=0}^T (p_\tau \cdot u_\tau + \tfrac{1}{2} u_\tau \cdot P_\tau u_\tau) + \sum_{\tau=1}^{T+1} (q_\tau \cdot v_\tau - \tfrac{1}{2} v_\tau \cdot Q_\tau v_\tau) \\ & - \sum_{\tau=1}^T v_\tau \cdot D_\tau u_\tau - [u, v], \end{aligned}$$

where  $[u, v]$  denotes the common value of the expression in (3.3).

Then  $U$  and  $V$  are polyhedral convex sets, and  $J$  is a quadratic convex-concave function.

**Proof.** This is immediate from our assumptions and the fact the expression  $[u, v]$  is affine in  $u$  and  $v$  separately.  $\square$

**Theorem 3.3.** The deterministic optimal control problems  $(\mathcal{P}_{\text{det}})$  and  $(\mathcal{Q}_{\text{det}})$  are the primal and dual problems of generalized linear-quadratic programming associated with the  $U$ ,  $V$ , and  $J$  in Proposition 3.2. In particular, the assertions of Theorem 2.1 are valid for  $(\mathcal{P}_{\text{det}})$  and  $(\mathcal{Q}_{\text{det}})$ .

**Proof.** We need only show that the expressions  $f(u)$  and  $g(v)$  in  $(\mathcal{P}_{\text{det}})$  and  $(\mathcal{Q}_{\text{det}})$  arise according to the pattern in the general problems  $(\mathcal{P})$  and  $(\mathcal{Q})$  of §1. First using for  $[u, v]$  in (3.4) the right hand expression in (3.3), we write

$$\begin{aligned}
 J(u, v) = & \sum_{\tau=0}^T (p_{\tau} \cdot u_{\tau} + \frac{1}{2} u_{\tau} \cdot P_{\tau} u_{\tau}) - \sum_{\tau=1}^{T+1} c_{\tau} \cdot x_{\tau-1} \\
 (3.5) \quad & + \sum_{\tau=1}^T ([q_{\tau} - C_{\tau} x_{\tau-1} - D_{\tau} u_{\tau}] \cdot v_{\tau} - \frac{1}{2} v_{\tau} \cdot Q_{\tau} v_{\tau}) \\
 & + ([q_{T+1} - C_{T+1} x_T] \cdot v_{T+1} - \frac{1}{2} v_{T+1} \cdot Q_{T+1} v_{T+1})
 \end{aligned}$$

The maximization of this over all  $v \in V$  reduces to a separate maximization with respect to each of the components  $v_{\tau}$  of  $v$ . Since by definition

$$\sup_{v_{\tau} \in V_{\tau}} \{ [q_{\tau} - C_{\tau} x_{\tau-1} - D_{\tau} u_{\tau}] \cdot v_{\tau} - \frac{1}{2} v_{\tau} \cdot Q_{\tau} v_{\tau} \} = \rho_{V_{\tau}, Q_{\tau}} (q_{\tau} - C_{\tau} x_{\tau-1} - D_{\tau} u_{\tau})$$

and

$$\sup_{v_{T+1} \in V_{T+1}} \{ [q_{T+1} - C_{T+1} x_T] \cdot v_{T+1} - \frac{1}{2} v_{T+1} \cdot Q_{T+1} v_{T+1} \} = \rho_{V_{T+1}, Q_{T+1}} (q_{T+1} - C_{T+1} x_T),$$

we conclude that  $\sup_{v \in V} J(u, v)$  is the  $f(u)$  in  $(\mathcal{P}_{\text{det}})$ .

Next using for  $[u, v]$  the left hand expression in (3.3), we write

$$\begin{aligned}
 J(u, v) = & \sum_{\tau=1}^{T+1} (q_{\tau} \cdot v_{\tau} - \frac{1}{2} v_{\tau} \cdot Q_{\tau} v_{\tau}) - \sum_{\tau=0}^T b_{\tau} \cdot y_{\tau+1} \\
 (3.6) \quad & - \sum_{\tau=1}^T ([B_{\tau}^* y_{\tau+1} + D_{\tau}^* v_{\tau} - p_{\tau}] \cdot u_{\tau} - \frac{1}{2} u_{\tau} \cdot P_{\tau} u_{\tau}) \\
 & - ([B_0^* y_1 - p_0] \cdot u_0 - \frac{1}{2} u_0 \cdot P_0 u_0).
 \end{aligned}$$

The minimization of this over all  $u \in U$  reduces similarly to a separate minimization with respect to each of the components  $u_\tau$ . We know that

$$\sup_{u_\tau \in U_\tau} \{[B_\tau^* y_{\tau+1} + D_\tau^* v_\tau - p_\tau] \cdot u_\tau - \frac{1}{2} u_\tau \cdot P_\tau u_\tau\} = \rho_{u_\tau, P_\tau}(B_\tau^* y_{\tau+1} + D_\tau^* v_\tau - p_\tau)$$

and

$$\sup_{u_0 \in U_0} \{[B_0^* y_1 - p_0] \cdot u_0 - \frac{1}{2} u_0 \cdot P_0 u_0\} = \rho_{u_0, P_0}(B_0^* y_1 - p_0).$$

We conclude that  $\inf_{u \in U} J(u, v)$  is the  $g(v)$  in  $(\mathcal{Q}_{\det})$ .  $\square$

The proof of Theorem 3.3 reveals an important simplifying feature of our minimax representation of  $(\mathcal{P}_{\det})$  and  $(\mathcal{Q}_{\det})$ . We state it as follows.

**Theorem 3.4.** *For the  $U$ ,  $V$ , and  $J$  in Theorem 3.3 one has the following decomposability properties for separate minimization in  $u$  or maximization in  $v$ . Here  $\bar{u}$  and  $\bar{v}$  are elements of  $U$  and  $V$ , and  $\bar{x}$  and  $\bar{y}$  the corresponding trajectories.*

(a)  $\tilde{u} \in \operatorname{argmin}_{u \in U} J(u, \bar{v})$  if and only if

$$\begin{aligned} \tilde{u}_\tau &\in \partial \rho_{U_\tau, P_\tau}(B_\tau^* \bar{y}_{\tau+1} + D_\tau^* \bar{v}_\tau - p_\tau) \\ &= \operatorname{argmax}_{u_\tau \in U_\tau} \{[B_\tau^* \bar{y}_{\tau+1} + D_\tau^* \bar{v}_\tau - p_\tau] \cdot u_\tau - \frac{1}{2} u_\tau \cdot P_\tau u_\tau\} \end{aligned}$$

for  $\tau = 1, \dots, T$ , and

$$\begin{aligned} \tilde{u}_0 &\in \partial \rho_{U_0, P_0}(B_0^* \bar{y}_1 - p_0) \\ &= \operatorname{argmax}_{u_0 \in U_0} \{[B_0^* \bar{y}_1 - p_0] \cdot u_0 - \frac{1}{2} u_0 \cdot P_0 u_0\}. \end{aligned}$$

(b)  $\tilde{v} \in \operatorname{argmax}_{v \in V} J(\bar{u}, v)$  if and only if

$$\begin{aligned} \tilde{v}_\tau &\in \partial \rho_{V_\tau, Q_\tau}(q_\tau - C_\tau \bar{x}_{\tau-1} - D_\tau \bar{u}_\tau) \\ &= \operatorname{argmax}_{v_\tau \in V_\tau} \{[q_\tau - C_\tau \bar{x}_{\tau-1} - D_\tau \bar{u}_\tau] \cdot v_\tau - \frac{1}{2} v_\tau \cdot Q_\tau v_\tau\} \end{aligned}$$

for  $\tau = 1, \dots, T$ , and

$$\begin{aligned} \tilde{v}_{T+1} &\in \partial \rho_{V_{T+1}, Q_{T+1}}(q_{T+1} - C_{T+1} \bar{x}_T) \\ &= \operatorname{argmax}_{v_{T+1} \in V_{T+1}} \{[q_{T+1} - C_{T+1} \bar{x}_T] \cdot v_{T+1} - \frac{1}{2} v_{T+1} \cdot Q_{T+1} v_{T+1}\}. \end{aligned}$$

**Proof.** The formulas in terms of “argmax” are justified by the calculations in the proof of Theorem 3.3. The question that remains is whether the “argmax” sets are truly the same as the indicated subgradient sets. This is answered by the observation that in the notation (2.2) one has  $\rho_{V, Q} = \theta_{V, Q}^*$  (convex conjugate), where

$$(3.7) \quad \theta_{V, Q}(v) = \begin{cases} \frac{1}{2} v \cdot Q v & \text{if } v \in V, \\ \infty & \text{if } v \notin V. \end{cases}$$

Inasmuch as  $\theta_{V,Q}$  is a closed proper convex function, one also has  $\theta_{V,Q} = \rho_{V,Q}^*$  and

$$\partial \rho_{V,Q}(r) = \operatorname{argmax}_{v \in \mathbb{R}^\ell} \{r \cdot v - \theta_{V,Q}(v)\}$$

by the basic rules of convex analysis [8, Theorem 12.2]. When this is applied to the pairs  $V_\tau, Q_\tau$ , and  $U_\tau, P_\tau$ , in place of  $V, Q$ , we reach our desired conclusion.  $\square$

The significance of the formulas in Theorem 3.4 lies in their potential use in iterative methods for solving  $(\mathcal{P}_{\text{det}})$  and  $(\mathcal{Q}_{\text{det}})$  when the dimensions

$$(3.9) \quad k = \sum_{\tau=0}^T k_\tau \text{ and } \ell = \sum_{\tau=1}^{T+1} \ell_\tau$$

of the vectors  $u = (u_0, u_1, \dots, u_T)$  and  $v = (v_1, \dots, v_T, v_{T+1})$  are large. The dimensions may be expected to be large if  $T$  is large, as of course would happen in particular in taking  $(\mathcal{P}_{\text{det}})$  and  $(\mathcal{Q}_{\text{det}})$  to be discrete-time approximations to continuous-time control problems such as the ones studied in [1]. In the presence of high dimensions, it may be impossible or inexpedient to solve  $(\mathcal{P}_{\text{det}})$  and  $(\mathcal{Q}_{\text{det}})$  directly by reducing them to ordinary quadratic programming problems in duality and applying a typical finitely-terminating quadratic programming code (as would be possible in principle in a manner explained in Rockafellar and Wets [3, §2]).

An alternative approach in that case is the exploration of methods that determine approximate solutions to  $(\mathcal{P}_{\text{det}})$  and  $(\mathcal{Q}_{\text{det}})$  by calculating a sequence of approximate saddlepoints  $(\bar{u}^\nu, \bar{v}^\nu)$  of  $J$  on  $U \times V$  for  $\nu = 1, 2, \dots$ , as suggested by the characterization of optimality in Theorem 3.4. In any such method the ability to calculate

$$(3.10) \quad f(\bar{u}^\nu) = \max_{v \in V} J(\bar{u}^\nu, v) \text{ and } \tilde{v}^\nu \in \operatorname{argmax}_{v \in V} J(\bar{u}^\nu, v)$$

as well as

$$(3.11) \quad g(\bar{v}^\nu) = \min_{u \in U} J(u, \bar{v}^\nu) \text{ and } \tilde{u}^\nu \in \operatorname{argmin}_{u \in U} J(u, \bar{v}^\nu)$$

is crucial in producing primal and dual bounds that tell how far  $\bar{u}^\nu$  and  $\bar{v}^\nu$  are from optimality and as input to possible schemes for updating  $(\bar{u}^\nu, \bar{v}^\nu)$  to  $(\bar{u}^{\nu+1}, \bar{v}^{\nu+1})$ . Theorem 3.4 says that the calculations in (3.10) and (3.11) can feasibly be carried out in terms of solving a collection of low-dimensional quadratic programming subproblems indexed by  $\tau$ . Moreover these subproblems can even be solved in “closed form”, i.e. without applying a quadratic programming code, if the functions  $\rho_{V_\tau, Q_\tau}$  and  $\rho_{U_\tau, P_\tau}$  have sufficiently simple expressions that allow the use of subgradient formulas directly.

The subgradient formulas are readily usable, for example, in the completely decomposable case where  $U_\tau$  and  $V_\tau$  are boxes (products of closed intervals, e.g. orthants) and  $P_\tau$  and  $Q_\tau$  are diagonal. Indeed, if  $P_\tau$  and  $Q_\tau$  are nonsingular the subgradients reduce to gradients given by very elementary expressions.

**Theorem 3.5.** Consider a control pair  $\bar{u}, \bar{v}$ , and the corresponding trajectories  $\bar{x}$  and  $\bar{y}$  determined by (3.1) and (3.2). Define

$$(3.12) \quad \bar{p}_\tau = p_\tau - B_\tau^* \bar{y}_{\tau+1} \text{ for } \tau = 0, 1, \dots, T, \text{ and } \bar{q}_\tau = q_\tau - C_\tau \bar{x}_{T-1} \text{ for } \tau = 1, \dots, T, T+1.$$

Let  $(\bar{P}_\tau)$  and  $(\bar{Q}_\tau)$  for  $\tau = 1, \dots, T$  denote the primal and dual problems of generalized linear-quadratic programming associated with

$$(3.13) \quad J_\tau(u_\tau, v_\tau) = \bar{p}_\tau \cdot u_\tau + \frac{1}{2} u_\tau \cdot P_\tau u_\tau + \bar{q}_\tau \cdot v_\tau - \frac{1}{2} v_\tau \cdot Q_\tau v_\tau - v_\tau \cdot D_\tau u_\tau$$

on  $U_\tau \times V_\tau$ , namely,

$$(\bar{P}_\tau) \quad \text{minimize } \bar{p}_\tau \cdot u_\tau + \frac{1}{2} u_\tau \cdot P_\tau u_\tau + \rho_{V_\tau, Q_\tau}(\bar{q}_\tau - D_\tau u_\tau) \text{ over } u_\tau \in U_\tau,$$

$$(\bar{Q}_\tau) \quad \text{maximize } \bar{q}_\tau \cdot v_\tau - \frac{1}{2} v_\tau \cdot Q_\tau v_\tau - \rho_{U_\tau, P_\tau}(D_\tau^* v_\tau - \bar{p}_\tau) \text{ over } v_\tau \in V_\tau,$$

and consider also the problems

$$(\bar{P}_0) \quad \text{minimize } \bar{p}_0 \cdot u_0 + \frac{1}{2} u_0 \cdot P_0 u_0 \text{ over } u_0 \in U_0,$$

$$(\bar{Q}_{T+1}) \quad \text{maximize } \bar{q}_{T+1} \cdot v_{T+1} - \frac{1}{2} v_{T+1} \cdot Q_{T+1} v_{T+1} \text{ over } v_{T+1} \in U_{T+1}.$$

Then a necessary and sufficient condition for  $\bar{u}$  and  $\bar{v}$  to be optimal solutions to the control problems  $(\mathcal{P}_{\det})$  and  $(\mathcal{Q}_{\det})$ , respectively, is that  $\bar{u}_\tau$  should be an optimal solution to the subproblem  $(\bar{P}_\tau)$  for  $\tau = 0, 1, \dots, T$ , and  $\bar{v}_\tau$  should be an optimal solution to the subproblem  $(\bar{Q}_\tau)$  for  $\tau = 1, \dots, T, T+1$ .

**Proof.** We know from Theorem 3.3 that a necessary and sufficient condition for the optimality of  $\bar{u}$  and  $\bar{v}$  in  $(\mathcal{P}_{\det})$  and  $(\mathcal{Q}_{\det})$  is the saddlepoint relation

$$\bar{u} \in \operatorname{argmin}_{u \in U} J(u, \bar{v}) \text{ and } \bar{v} \in \operatorname{argmax}_{v \in V} J(\bar{u}, v).$$

Furthermore, this reduces to having the argmax conditions in Theorem 3.4 hold for  $\tilde{u} = \bar{u}$  and  $\tilde{v} = \bar{v}$ . These conditions in turn are equivalent to

$$\begin{aligned}\bar{u}_\tau &\in \operatorname{argmin}_{u_\tau \in U_\tau} J_\tau(u_\tau, (\bar{v})_\tau) \text{ for } \tau = 1, \dots, T, \\ \bar{u}_0 &\in \operatorname{argmin}_{u_0 \in U_0} \{\bar{p}_0 \cdot u_0 + \frac{1}{2} u_0 \cdot P_0 u_0\},\end{aligned}$$

and

$$\begin{aligned}\bar{v}_\tau &\in \operatorname{argmax}_{v_\tau \in V_\tau} J_\tau(\bar{u}_\tau, v_\tau) \text{ for } \tau = 1, \dots, T, \\ \bar{v}_{T+1} &\in \operatorname{argmax}_{v_{T+1} \in V_{T+1}} \{\bar{q}_{T+1} \cdot v_{T+1} - \frac{1}{2} v_{T+1} \cdot Q_{T+1} v_{T+1}\}.\end{aligned}$$

The latter mean that  $\bar{u}_0$  is optimal for  $(P_0)$ ,  $\bar{v}_{T+1}$  is optimal for  $(Q_{T+1})$ , and  $(\bar{u}_\tau, \bar{v}_\tau)$  is a saddlepoint of  $J_\tau(u_\tau, v_\tau)$  relative to  $u_\tau \in U_\tau$  and  $v_\tau \in V_\tau$  for  $\tau = 1, \dots, T$ . This saddlepoint condition is equivalent by Theorem 2.1 to  $\bar{u}_\tau$  and  $\bar{v}_\tau$  being optimal solutions to the primal and dual subproblems  $(\bar{P}_\tau)$  and  $(\bar{Q}_\tau)$ .  $\square$

Optimality conditions of the kind in Theorem 3.5 were developed for continuous-time problems in Rockafellar [1]. They resemble conditions first detected in a special setting known as “continuous linear programming” by Grinold [9].

Besides being of interest in the study of what optimality might mean in a particular application modeled directly in terms of  $(P_{\det})$  and  $(Q_{\det})$ , the conditions in Theorem 3.5, like those in Theorem 3.4, have import for computations. Having arrived at a control pair  $(\bar{u}^\nu, \bar{v}^\nu)$  and associated trajectories  $(\bar{x}^\nu, \bar{y}^\nu)$  in some iteration  $\nu$  of a numerical method, one can construct a new pair  $(u^\nu, v^\nu) \in U \times V$  by taking  $u^\nu_\tau$  to be an optimal solution to  $(\bar{P}^\nu_\tau)$  for  $\tau = 0, 1, \dots, T$  and  $v^\nu_\tau$  an optimal solution to  $(\bar{Q}^\nu_\tau)$  for  $\tau = 1, \dots, T, T+1$ , where  $(\bar{P}^\nu_\tau)$  and  $(\bar{Q}^\nu_\tau)$  are the subproblems corresponding to  $\bar{u}^\nu$  and  $\bar{v}^\nu$  in the sense of Theorem 3.5. Then  $u^\nu$  and  $v^\nu$  generate new trajectories  $x^\nu$  and  $y^\nu$  that may be compared with  $\bar{x}^\nu$  and  $\bar{y}^\nu$ , and for so forth. This procedure, like the one described after Theorem 3.4, provides another tool that might be used constructively in the generation of a sequence of approximate saddlepoints.



#### 4. Stochastic Control Model.

The probability space we work with in this paper is simply a finite set  $\Omega$ , for reasons given in §1. The probability associated with an element  $\omega \in \Omega$  is  $\pi_\omega \geq 0$ ; one has  $\sum_{\omega \in \Omega} \pi_\omega = 1$ . The vectors, matrices and sets introduced in the formulation of our deterministic problems persist notationally in the stochastic problems, but all are now treated as (potentially) random variables. Thus, for example,  $p_\tau$  now denotes a mapping  $\omega \mapsto p_{\omega\tau} \in \mathbb{R}^{k_\tau}$  rather than necessarily just a single vector. Likewise  $P_\tau$  is a matrix-valued mapping  $\omega \mapsto P_{\omega\tau}$ , and  $U_\tau$  is a set-valued mapping  $\omega \mapsto U_{\omega\tau}$ . In line with our earlier assumptions, we suppose that  $P_{\omega\tau}$  and  $Q_{\omega\tau}$  are *positive semidefinite* (symmetric), and  $U_{\omega\tau}$  and  $V_{\omega\tau}$  are *polyhedral convex* (nonempty). The expectation of a random variable such as  $p_\tau$  is

$$E\{p_\tau\} = E_\omega\{p_{\omega\tau}\} := \sum_{\omega \in \Omega} \pi_\omega p_{\omega\tau}.$$

The information available to the decision-making process at time  $\tau$  is modeled by the specification of a (finite) field  $\mathcal{G}_\tau$  of subsets of  $\Omega$  for  $\tau = 0, 1, \dots, T, T+1$ . The fields  $\mathcal{G}_\tau$  may differ from the complete information fields  $\mathcal{F}_\tau$ , and no particular relation between them is presupposed, although the case where  $\mathcal{G}_\tau$ 's are increasing with  $\mathcal{G}_\tau$  contained in  $\mathcal{F}_\tau$  is, for instance, an important one. More will be said about this after the statement of our primal and dual problems. We assume that

$$(4.1) \quad U_\tau, V_\tau, p_\tau, P_\tau, q_\tau, Q_\tau, \text{ and } D_\tau \text{ are } \mathcal{G}_\tau\text{-measurable,}$$

but in general do *not* place this restriction on  $A_\tau, B_\tau, C_\tau, b_\tau$  or  $c_\tau$ . Trivially the latter are measurable with respect to the underlying field  $\mathcal{F}$  of complete information, comprised here of *all* the subsets of  $\Omega$ .

Because  $\mathcal{G}_\tau$  is a finite collection of subsets of  $\Omega$ , the notion of  $\mathcal{G}_\tau$ -measurability has an especially simple representation for our purposes. Let  $\mathcal{A}_\tau$  denote the subcollection of  $\mathcal{G}_\tau$  consisting of all  $\mathcal{G}_\tau$ -atoms, i.e. nonempty  $\mathcal{G}_\tau$ -measurable sets that do not properly include any other nonempty  $\mathcal{G}_\tau$ -measurable set. Such atoms are mutually disjoint. A set is  $\mathcal{G}_\tau$ -measurable if and only if it is a union of  $\mathcal{G}_\tau$ -atoms. Thus there is a one-to-one correspondence between  $\mathcal{G}_\tau$ -measurable sets in  $\Omega$  and sets of  $\mathcal{G}_\tau$ -atoms, i.e. subsets of  $\mathcal{A}_\tau$ . A function is  $\mathcal{G}_\tau$ -measurable if and only if it is constant relative to every  $\mathcal{G}_\tau$ -atom. Each  $\mathcal{G}_\tau$ -measurable function can in this way be identified uniquely with a function on  $\mathcal{A}_\tau$  rather than on  $\Omega$ . We can indicate this notationally, when we wish to, by writing  $p_{\alpha\tau}$  for  $\alpha \in \mathcal{A}_\tau$  to denote the common value that  $p_{\omega\tau}$  has for all  $\omega \in \alpha$  when  $p$  is  $\mathcal{G}_\tau$ -measurable. (Obviously  $\Omega$  itself in this setting might be identified with the set of atoms of some finite field of

information chosen within a larger, possibly “continuous” probability space by some kind of approximation. We don’t go into this matter here.)

Conditional expectation with respect to  $\mathcal{G}_\tau$  is denoted by  $E^{\mathcal{G}_\tau}$ . This can be viewed in the present setting as the linear transformation that takes a random variable such as  $B_\tau$  and redefines it to have a constant value on each  $\mathcal{G}_\tau$ -atom  $\alpha \in \mathcal{A}_\tau$ , that value being, of course, the “weighted average”

$$\left[ \sum_{\omega \in \alpha} \pi_\omega B_{\omega\tau} \right] / \left[ \sum_{\omega \in \alpha} \pi_\omega \right].$$

The stochastic dynamical systems for our primal and dual problems are taken again to have the forms (3.1) and (3.2), but with all elements now interpreted as (potentially) random, and with the restriction that

$$(4.2) \quad u_\tau \text{ is } \mathcal{G}_\tau - \text{measurable,}$$

$$(4.3) \quad v_\tau \text{ is } \mathcal{G}_\tau - \text{measurable.}$$

The condition  $u_\tau \in U_\tau$  in (3.1) is interpreted to mean that  $u_{\omega\tau} \in U_{\omega\tau}$  for all  $\omega \in \Omega$ , and similarly for  $v_\tau \in V_\tau$ . Our primal problem of stochastic control is

$$\begin{aligned} & \text{minimize subject to (3.1) and (4.2) the function } f(u) = \\ & \sum_{\tau=0}^T E\{p_\tau \cdot u_\tau + \tfrac{1}{2} u_\tau \cdot P_\tau u_\tau\} - \sum_{\tau=1}^{T+1} E\{c_\tau \cdot x_{\tau-1}\} \\ (P_{\text{sto}}) \quad & + \sum_{\tau=1}^T E\{\rho_{V_\tau, Q_\tau}(q_\tau - E^{\mathcal{G}_\tau}\{C_\tau x_{\tau-1}\} - D_\tau u_\tau)\} \\ & + E\{\rho_{V_{T+1}, Q_{T+1}}(q_{T+1} - E^{\mathcal{G}_{T+1}}\{C_{T+1} x_T\})\}. \end{aligned}$$

The corresponding dual problem is

$$\begin{aligned} & \text{maximize subject to (3.2) and (4.3) the function } g(v) = \\ (Q_{\text{sto}}) \quad & \sum_{\tau=1}^{T+1} E\{q_\tau \cdot v_\tau - \tfrac{1}{2} v_\tau \cdot Q_\tau v_\tau\} - \sum_{\tau=1}^T E\{b_\tau \cdot y_{\tau+1}\} \\ & - \sum_{\tau=1}^T E\{\rho_{U_\tau, P_\tau}(E^{\mathcal{G}_\tau}\{B_\tau^* y_{\tau+1}\} + D_\tau^* v_\tau - p_\tau)\} - E\{\rho_{U_0, P_0}(E^{\mathcal{G}_0}\{B_0^* y_1 - p_0\})\}. \end{aligned}$$

Here  $\rho_{V_\tau, Q_\tau}$  and  $\rho_{U_\tau, P_\tau}$  are “random functions” that depend  $\mathcal{G}_\tau$ -measurably on  $\omega \in \Omega$  by virtue of (4.1). The random variables

$$\xi_\tau := E^{\mathcal{G}_\tau}\{C_\tau x_{\tau-1}\} \text{ and } \eta_\tau := E^{\mathcal{G}_\tau}\{B_\tau^* y_{\tau+1}\}$$

and  $\mathcal{G}_\tau$ -measurable too, of course, so the arguments to which  $\rho_{V_\tau, Q_\tau}$  and  $\rho_{U_\tau, P_\tau}$  are applied are always  $\mathcal{G}_\tau$ -measurable. The  $\rho$  terms at time  $\tau$  thus monitor “constraint expressions” based solely on the information available to the decision maker available at time  $\tau$ . Note from the dynamics that  $\xi_{\omega\tau}$  depends affinely on  $u_{\omega 0}, \dots, u_{\omega, \tau-1}$ , whereas  $\eta_{\omega\tau}$  depends affinely on  $v_{\omega, \tau+1}, \dots, v_{\omega, T+1}$ .

Although in the formulation of our stochastic control problem ( $\mathcal{P}_{\text{sto}}$ ) the information fields  $\mathcal{G}_\tau$  are independent of the earlier controls  $(u_0, \dots, u_{\tau-1})$ , this does not mean that the observations to which we have access are independent of  $(u_0, \dots, u_{\tau-1})$ . In fact, quite often the “raw” information available at time  $\tau$ , consists of a collection of vectors  $(z_0, \dots, z_{\tau-1})$  that represent either complete or partial observations of the past states of the system  $(x_0, x_1, \dots, x_{\tau-1})$ . These observations may even be corrupted by measurement noise. The “classical” formulation of the stochastic control problem, as in [10] for example, defines the current information field in terms of the field  $S_\tau$  generated by these observations. To fix the ideas, suppose that the parameters of the objective are not stochastic, that the dynamics of the control problem are given by (3.1) and that the observation  $z_\tau$  at time  $\tau$  is a function of the state of the system given by

$$z_\tau = H_\tau x_\tau + h_\tau$$

where the matrix  $H_\tau$  is  $m_\tau \times n_\tau$  and  $h_\tau$  is a random  $m_\tau$ -vector which, in order to stay in the present framework of discrete probability is assumed to have a discrete distribution. From (3.1) it follows that

$$z_\tau = \sum_{r=0}^{\tau} \left( \prod_{k=r+1}^{\tau} A_k \right) (B_r u_r + b_r)$$

(with the convention that the empty product  $\prod_{k=r+1}^{\tau} A_k$  is  $I$ ), and thus

$$z_\tau = \sum_{t=0}^{\tau} H_\tau \left( \prod_{k=t+1}^{\tau} A_k \right) B_t u_t + \sum_{t=0}^{\tau} H_\tau \left( \prod_{k=t+1}^{\tau} A_k \right) b_t + h_t.$$

Once the values of  $(u_0, u_1, \dots, u_{\tau-1})$  are fixed, the field  $S_\tau$  generated by the random variables  $(z_0, z_1, \dots, z_{\tau-1})$  can be derived from the field generated by the stochastic elements of

$$(A_1, \dots, A_{\tau-1}, B_0, \dots, B_{\tau-1}, b_0, \dots, b_{\tau-1}, h_0, \dots, h_{\tau-1}).$$

But also a converse of sorts does hold. When the matrices  $A_t$  and  $B_t$  are nonstochastic, or more generally, the values of  $A_\tau$  and  $B_\tau$  will be known at time  $\tau$ , then from the observations

$(z_0, \dots, z_{\tau-1})$  and the controls  $(u_0, \dots, u_{\tau-1})$  it is possible to re-express the information in terms of a field  $\mathcal{G}_\tau$  defined on the support of the random vectors  $(b_0, b_1, \dots, b_{\tau-1})$  that does not depend on the control variables. Indeed, in this case with

$$G_{k,t} = H_k \left( \prod_{i=t+1}^k A_i \right), \quad \text{for } t = 0, \dots, k,$$

and

$$g_k = \left( \sum_{t=0}^k G_{k,t} B_t u_t \right) + h_k,$$

we have the linear system

$$z_k = g_k + G_{k,0}b_0 + \dots + G_{k,k}b_k, \quad k = 0, \dots, \tau - 1.$$

From these relations, it follows that every value taken on by the random variables  $(z_0, \dots, z_{\tau-1})$ , that depend on the controls (via the random vectors  $g_0, \dots, g_{\tau-1}$ , determines a set of possible values for the random variables  $h_0, \dots, h_{\tau-1}$ ,  $b_0, \dots, b_{\tau-1}$ ). The projection of these sets on the support of the random vectors  $(b_0, \dots, b_{\tau-1})$  engenders the atoms of  $\mathcal{G}_\tau$ .

What this shows is that our model does include a much richer class of stochastic control problems as might appear to be the case at first. The example (of a problem with noisy partial observations of the state) is by no means the only "extension". From the preceding derivation it is clear that we can even allow for certain nonlinearities in the relation between state and observation, that the condition of full knowledge of the matrices  $A_\tau$  and  $B_\tau$  at time  $\tau$  can be relaxed in certain cases, and so on. We favor the formulation of the stochastic control problem in terms of the  $\mathcal{G}_\tau$ -measurability of the controls, although it may sometimes seem simpler (and more appropriate) to express the dependence of the controls on the available information in terms of the field  $\mathcal{S}_\tau$  generated by the observations, because the resulting structure is directly amenable to the use of linear-quadratic programming techniques. And from a computational viewpoint these go much beyond the capabilities of standard dynamic programming procedures, as will be clear from the results that follow.

Before we return to the characterization of optimal controls and trajectories, let us also note that because we allow the dimensionality of the state and control vectors to vary over time, our model also includes the classical multistage recourse models. Suppose that the equations (3.1) have the special form

$$\begin{aligned} x_\tau &= \begin{bmatrix} I \\ 0 \end{bmatrix} x_{\tau-1} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_\tau \quad \text{for } \tau = 1, \dots, T, \\ x_0 &= u_0, \end{aligned}$$

where the identity matrices  $I$  and zero matrices  $0$  are of the appropriate dimensions. Then

$$x_0 = u_0, \quad x_1 = (u_0, u_1)^T, \quad x_2 = (u_0, u_1, u_2)^T, \text{ etc.}$$

Thus  $x_\tau$  is the “memory” of all decisions up through time  $\tau$ . Assuming that  $\mathcal{G}_\tau = \mathcal{F}_\tau$  (complete information field), we get  $x_\tau$ , like  $u_\tau$  to be  $\mathcal{F}_\tau$ -measurable. Then in  $(\mathcal{P}_{\text{sto}})$  the term

$$q_\tau - E^{\mathcal{F}_\tau} \{C_\tau x_{\tau-1}\} - D_\tau u_\tau$$

represents a general affine expression in  $u_0, u_1, \dots, u_\tau$ . When  $\rho_{V_\tau, Q_\tau}$  is of the type (2.4), we can rewrite  $(\mathcal{P}_{\text{sto}})$  in terms of linear constraints and a quadratic objective involving only the control variables  $u_0, u_1, \dots, u_T$ . This problem, with its block angular structure, is in the usual format of the multistage stochastic program with recourse model, see [11] or [12], for example.

Problem  $(\mathcal{P}_{\text{sto}})$  revolves around the choice of the random variable  $u = (u_0, u_1, \dots, u_T)$ , which can be regarded as a function from  $\Omega$  to  $\mathbb{R}^{k_0} \times \dots \times \mathbb{R}^{k_T}$  and therefore as an element of the finite-dimensional vector space consisting of all such functions. The dimension of this space may be very large indeed just from the size of  $\Omega$  and possibly  $T$ , even if  $k_0, \dots, k_T$  are themselves relatively small, as might generally be supposed. We must therefore think of  $(\mathcal{P}_{\text{sto}})$  as inherently a “large-scale” problem for which approximate methods of solution will be more appropriate than “exact” ones.

Nevertheless it is well to keep in mind that the representation of  $u$  as a function from  $\Omega$  to  $\mathbb{R}^{k_0} \times \dots \times \mathbb{R}^{k_T}$  tends to exaggerate the dimensionality of  $(\mathcal{P}_{\text{sto}})$ . The constraint that  $u_\tau$  be  $\mathcal{G}_\tau$ -measurable means, as already noted, that  $u_\tau$  can be identified uniquely with a certain function from  $\mathcal{A}_\tau$  to  $\mathbb{R}^{k_\tau}$ . The dimension of the space of all functions from  $\mathcal{A}_\tau$  to  $\mathbb{R}^{k_\tau}$  is  $a_\tau k_\tau$ , where

$$a_k = |\mathcal{A}_k| \text{ (the number of atoms in } \mathcal{G}_k \text{)}.$$

Thus the “true” dimensionality of  $(\mathcal{P}_{\text{sto}})$ , in the sense of the number of real-valued decision variables, is

$$(4.5) \quad k^* = a_0 k_0 + a_1 k_1 + \dots + a_T k_T.$$

By the same token, the “true” dimensionality of  $(\mathcal{Q}_{\text{sto}})$ , where the random variable  $v = (v_1, \dots, v_T, v_{T+1})$  must be optimized, is

$$(4.6) \quad \ell^* = a_1 \ell_1 + \dots + a_T \ell_T + a_{T+1} \ell_{T+1}.$$

**Proposition 4.1.** *Let*

$$\begin{aligned}\mathcal{U} &= \{u = (u_0, u_1, \dots, u_T) \mid u_\tau \text{ is } \mathcal{G}_\tau\text{-measurable with } u_\tau \in U_\tau\}, \\ \mathcal{V} &= \{v = (v_1, \dots, v_T, v_{T+1}) \mid v_\tau \text{ is } \mathcal{G}_\tau\text{-measurable with } v_\tau \in V_\tau\},\end{aligned}$$

and define  $J(u, v) = E\{J(u, v)\}$ , where  $J(u, v)$  is the expression in Proposition 3.2 (regarded now as a random variable depending on the choice of the random variables  $u$  and  $v$ ). Then  $\mathcal{U}$  and  $\mathcal{V}$  are polyhedral convex sets (nonempty), and  $J$  is a quadratic convex-concave function.

**Proof.** By definition  $\mathcal{U}$  is a subset of the space of all functions from  $\Omega$  to  $\mathbb{R}^{k_0} \times \dots \times \mathbb{R}^{k_T}$  consisting of the functions  $u$  such that  $u_{\omega\tau} \in U_{\omega\tau}$  for all  $\omega$  and  $\tau$ , and  $U_{\omega\tau}$  is constant in  $\omega$  with respect to each  $\mathcal{G}_\tau$ -atom  $\alpha \in \mathcal{A}_\tau$ . These conditions can be represented by a finite system of linear equations and inequalities, because  $\Omega$  is finite and  $U_{\omega\tau}$  is by assumption a convex polyhedron for each  $\omega$  and  $\tau$ . (Alternatively  $\mathcal{U}$  can be viewed as a direct product of polyhedral convex sets  $U_{\alpha\tau}$  indexed by  $\alpha \in \mathcal{A}_\tau$  and  $\tau = 0, 1, \dots, T$ , inasmuch as  $U_\tau$  is  $\mathcal{G}_\tau$ -measurable.) Thus  $\mathcal{U}$  is a convex polyhedron. Similarly  $\mathcal{V}$  is a convex polyhedron. We have by definition

$$J(u, v) = \sum_{\omega \in \Omega} \pi_\omega J(u_{\omega 0}, u_{\omega 1}, \dots, u_{\omega T}; v_{\omega 1}, \dots, v_{\omega T}, v_{\omega, T+1})$$

where the  $J$  term for each  $\omega$  is quadratic convex-concave function and the coefficients  $\pi_\omega$  are nonnegative therefore  $J$  is a quadratic convex-concave function.  $\square$

**Theorem 4.2.** *The stochastic optimal control problems  $(\mathcal{P}_{\text{sto}})$  and  $(\mathcal{Q}_{\text{sto}})$  are the primal and dual problems of generalized linear-quadratic programming associated with the  $\mathcal{U}$ ,  $\mathcal{V}$  and  $J$  in Proposition 4.1. In particular, the assertions of Theorem 2.1 are valid for  $(\mathcal{P}_{\text{sto}})$  and  $(\mathcal{Q}_{\text{sto}})$ .*

**Proof.** We must show that the supremum of  $J(u, v)$  over all  $v \in \mathcal{V}$  is the function  $f(u)$  in  $(\mathcal{P}_{\text{sto}})$ , and the infimum of  $J(u, v)$  over all  $u \in \mathcal{U}$  is  $g(u)$  in  $(\mathcal{Q}_{\text{sto}})$ . Starting with  $J(u, v)$  in the form of (3.5) (which is obtained by using the right hand expression in (3.3) for  $[u, v]$ ) and taking the expectation, we get by (4.1) that

$$\begin{aligned}J(u, v) &= \sum_{\tau=0}^T E\{p_\tau \cdot u_\tau + \frac{1}{2}u_\tau \cdot P_\tau u_\tau\} - \sum_{\tau=1}^{T+1} E\{c_\tau \cdot x_{\tau-1}\} \\ &\quad + \sum_{\tau=1}^T E\{[q_\tau - E^{\mathcal{G}_\tau}\{C_\tau x_{\tau-1}\} - D_\tau u_\tau] \cdot v_\tau - \frac{1}{2}v_\tau \cdot Q_\tau v_\tau\} \\ &\quad + E\{[q_{T+1} - E^{\mathcal{G}_{T+1}}\{C_{T+1} x_T\}] \cdot v_{T+1} - \frac{1}{2}v_{T+1} \cdot Q_{T+1} v_{T+1}\}\end{aligned}$$

To maximize this over all  $v \in \mathcal{V}$ , we must maximize separately in each of the  $v_\tau$ 's subject to  $v_\tau$  being a  $\mathcal{G}_\tau$ -measurable function with  $v_\tau \in V_\tau$ . Denote the random variable  $q_\tau - E^{\mathcal{G}_\tau}\{c_\tau x_\tau\} - D_\tau u_\tau$  temporarily by  $r_\tau$  for  $\tau = 1, \dots, T$  and  $q_{T+1} - E^{\mathcal{G}_{T+1}}\{C_{T+1} x_T\}$  by  $r_{T+1}$ . Then each  $r_\tau$  is  $\mathcal{G}_\tau$ -measurable and

$$J(u, v) = \sum_{\tau=0}^T E\{p_\tau u_\tau + \frac{1}{2} u_\tau \cdot P_\tau u_\tau\} - \sum_{\tau=1}^{T+1} E\{c_\tau \cdot x_{\tau-1}\} \\ \sum_{\tau=0}^T \sup_{v_\tau \in \mathcal{V}_\tau} E\{r_\tau \cdot v_\tau - \frac{1}{2} v_\tau \cdot Q_\tau v_\tau\},$$

where  $\mathcal{V}_\tau$  is the set of all  $\mathcal{G}_\tau$ -measurable  $v_\tau$  with  $v_\tau \in V_\tau$ . Since  $\mathcal{G}_\tau$ -measurable functions can be indexed by  $\alpha \in \mathcal{A}_\tau$  in place of  $\omega \in \Omega$ , as explained above, we can write

$$E\{r_\tau \cdot v_\tau - \frac{1}{2} v_\tau \cdot Q_\tau v_\tau\} = \sum_{\alpha \in \mathcal{A}_\tau} \pi_\alpha [r_{\alpha\tau} \cdot v_{\alpha\tau} - \frac{1}{2} v_{\alpha\tau} \cdot Q_{\alpha\tau} v_{\alpha\tau}],$$

where  $\pi_\alpha$  is the probability of the atom  $\alpha$ , i.e.

$$\pi_\alpha = \sum_{\omega \in \alpha} \pi_\omega.$$

The supremum of this expression over all  $v_\tau \in \mathcal{V}_\tau$  is

$$\sum_{\alpha \in \mathcal{A}_\tau} \pi_\alpha \sup_{v_{\alpha\tau} \in V_{\alpha\tau}} \{r_{\alpha\tau} \cdot v_{\alpha\tau} - \frac{1}{2} v_{\alpha\tau} \cdot Q_{\alpha\tau} v_{\alpha\tau}\} \\ = \sum_{\alpha \in \mathcal{A}_\tau} \pi_\alpha \rho_{V_{\alpha\tau}, Q_{\alpha\tau}}(r_{\alpha\tau}) = E\{\rho_{V_\tau, Q_\tau}(r_\tau)\}.$$

Thus the supremum of  $J(u, v)$  over  $v \in \mathcal{V}$  is

$$\sum_{\tau=0}^T E\{p_\tau \cdot u_\tau - \frac{1}{2} u_\tau \cdot P_\tau u_\tau\} - \sum_{\tau=1}^T E\{c_\tau \cdot x_{\tau-1}\} + \sum_{\tau=1}^{T+1} E\{\rho_{V_\tau, Q_\tau}(r_\tau)\},$$

which from choice of the  $r_\tau$ 's is the objective  $f(u)$  in  $(\mathcal{P}_{\text{sto}})$ . The argument that the infimum of  $J(u, v)$  over  $u \in \mathcal{U}$  is  $g(v)$  in  $(\mathcal{Q}_{\text{sto}})$  follows the same lines.  $\square$

**Theorem 4.3.** *For the  $\mathcal{U}$ ,  $\mathcal{V}$ , and  $J$  in Theorem 4.2 one has the following decomposability properties for separate minimization in  $u$  or maximization in  $v$ . The notation is used that*

$$\bar{v}_\tau = q_\tau - E^{\mathcal{G}_\tau}\{C_\tau \bar{x}_{\tau-1}\} - D_\tau \bar{u}_\tau \text{ for } \tau = 1, \dots, T, \\ \bar{r}_{T+1} = q_{T+1} - E^{\mathcal{G}_{T+1}}\{C_{T+1} \bar{x}_T\}, \\ \bar{s}_\tau = E^{\mathcal{G}_\tau}\{B_\tau^* \bar{y}_{\tau+1}\} + D_\tau^* \bar{v}_\tau - p_\tau \text{ for } \tau = 1, \dots, T, \\ \bar{s}_0 = E^{\mathcal{G}_0}\{B_0^* \bar{y}_1\} - p_0,$$

where  $\bar{u}$  and  $\bar{v}$  are elements of  $\mathcal{U}$  and  $\mathcal{V}$ , and  $\bar{x}$  and  $\bar{y}$  are the corresponding trajectories.

(a)  $\bar{u} \in \operatorname{argmin}_{u \in \mathcal{U}} J(u, v)$  if and only if

$$\bar{u}_{\alpha\tau} \in \partial \rho_{U_{\alpha\tau}, P_{\alpha\tau}}(\bar{s}_{\alpha\tau}) = \operatorname{argmax}_{u_{\alpha\tau} \in U_{\alpha\tau}} \{\bar{s}_{\alpha\tau} \cdot u_{\alpha\tau} - \frac{1}{2} u_{\alpha\tau} \cdot P_{\alpha\tau} u_{\alpha\tau}\}$$

for  $\tau = 0, 1, \dots, T$  and all  $\alpha \in \mathcal{A}_\tau$ .

(b)  $\bar{v} \in \operatorname{argmin}_{v \in \mathcal{V}} J(u, v)$  if and only if

$$\bar{v}_{\alpha\tau} \in \partial \rho_{V_{\alpha\tau}, Q_{\alpha\tau}}(\bar{r}_{\alpha\tau}) = \operatorname{argmax}_{v_{\alpha\tau} \in V_{\alpha\tau}} \{\bar{r}_{\alpha\tau} \cdot v_{\alpha\tau} - \frac{1}{2} v_{\alpha\tau} \cdot Q_{\alpha\tau} v_{\alpha\tau}\}$$

for  $\tau = 1, \dots, T, T+1$  and all  $\alpha \in \mathcal{A}_\tau$ .

**Proof.** This combines the argument of Theorem 4.2 with the conjugacy facts noted in the proof of Theorem 3.4.  $\square$

**Theorem 4.4.** Consider  $\mathcal{G}_\tau$ -measurable  $\bar{u}, \bar{v}$ , and the corresponding trajectories  $\bar{x}$  and  $\bar{y}$  determined by (3.1) and (3.2). Define the  $\mathcal{G}_\tau$ -measurable random variables

$$\begin{aligned} \bar{p}_\tau &= p_\tau - E^{\mathcal{G}_\tau} \{B_\tau^* \bar{y}_{\tau+1}\} \quad \text{for } \tau = 0, 1, \dots, T, \\ \bar{q}_\tau &= q_\tau - E^{\mathcal{G}_\tau} \{C_\tau \bar{x}_{\tau-1}\} \quad \text{for } \tau = 1, \dots, T, T+1. \end{aligned}$$

For each  $\tau = 1, \dots, T$  and  $\alpha \in \mathcal{A}_\tau$  let  $(\bar{P}_{\alpha\tau})$  and  $(\bar{Q}_{\alpha\tau})$  denote the primal and dual problems of generalized linear-quadratic programming associated with

$$J_{\alpha\tau}(u_{\alpha\tau}, v_{\alpha\tau}) = \bar{p}_{\alpha\tau} \cdot u_{\alpha\tau} + \frac{1}{2} u_{\alpha\tau} \cdot P_{\alpha\tau} u_{\alpha\tau} + \bar{q}_{\alpha\tau} v_{\alpha\tau} - \frac{1}{2} v_{\alpha\tau} \cdot Q_{\alpha\tau} v_{\alpha\tau} - v_{\alpha\tau} \cdot D_{\alpha\tau} u_{\alpha\tau}$$

on  $U_{\alpha\tau} \times V_{\alpha\tau}$ , namely

$$(\bar{P}_{\alpha\tau}) \quad \text{minimize } \bar{p}_{\alpha\tau} \cdot u_{\alpha\tau} + \frac{1}{2} u_{\alpha\tau} \cdot P_{\alpha\tau} u_{\alpha\tau} + \rho_{V_{\alpha\tau}, Q_{\alpha\tau}}(\bar{q}_{\alpha\tau} - D_{\alpha\tau} u_{\alpha\tau}) \text{ over } u_{\alpha\tau} \in U_{\alpha\tau},$$

$$(\bar{Q}_{\alpha\tau}) \quad \text{maximize } \bar{q}_{\alpha\tau} \cdot v_{\alpha\tau} - \frac{1}{2} v_{\alpha\tau} \cdot Q_{\alpha\tau} v_{\alpha\tau} - \rho_{U_{\alpha\tau}, P_{\alpha\tau}}(D_{\alpha\tau}^* v_{\alpha\tau} - \bar{p}_{\alpha\tau}) \text{ over } v_{\alpha\tau} \in V_{\alpha\tau},$$

and consider also the problems

$$(\bar{P}_{\alpha 0}) \quad \text{minimize } \bar{p}_{\alpha 0} \cdot u_{\alpha 0} + \frac{1}{2} u_{\alpha 0} \cdot P_{\alpha 0} u_{\alpha 0} \text{ over } u_{\alpha 0} \in U_{\alpha 0}$$

for  $\alpha \in \mathcal{A}_0$ , and

$$(\bar{Q}_{\alpha, T+1}) \quad \text{maximize } \bar{q}_{\alpha, T+1} \cdot u_{\alpha, T+1} - \frac{1}{2} u_{\alpha, T+1} \cdot P_{\alpha, T+1} u_{\alpha, T+1} \text{ over } u_{\alpha, T+1} \in U_{\alpha, T+1}$$

for  $\alpha \in \mathcal{A}_{T+1}$ .

Then a necessary and sufficient condition for  $\bar{u}$  and  $\bar{v}$  to be optimal solutions to the control problems  $(P_{\text{sto}})$  and  $(Q_{\text{sto}})$ , respectively, is that  $\bar{u}_{\alpha\tau}$  should be an optimal solution to the subproblem  $(\bar{P}_{\alpha\tau})$  for every  $\alpha \in \mathcal{A}_\tau$  and  $\tau = 0, 1, \dots, T$ , and  $\bar{v}_{\alpha\tau}$  should be an optimal solution to the subproblem  $(\bar{Q}_{\alpha\tau})$  for every  $\alpha \in \mathcal{A}_\tau$  and  $\tau = 1, \dots, T, T+1$ .

**Proof.** The argument imitates the one for Theorem 3.5 but uses the relations in Theorem 4.3.  $\square$



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