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PROOF FOR A CASE WHERE DISCOUNTING

ADVANCES THE DOOMSDAY

by Tjalling C. Koopmans*

In a previous paper (Koopmans [1973]), I considered some problems of "optimal" consumption \hat{p}_t over time of an exhaustible resource of known finite total availability R . In one of the cases studied, consumption of a minimum amount of the resource is assumed to be essential to human life, in such a way that all life ceases upon its exhaustion at time T . Assuming a constant population until that time, and denoting by \underline{r} the positive minimum consumption level needed for survival of that population, the survival period T is constrained by

$$(1) \quad 0 < T \leq R/\underline{r} \equiv \bar{T} .$$

Here equality ($T=\bar{T}$) can be attained only by consuming at the minimum level ($r_t=\underline{r}$) at all times, $0 \leq t \leq \bar{T}$.

However, optimality is defined in terms of maximization of the integral over time of discounted future utility levels,

$$(2) \quad V(\rho, T, (r_t)) \equiv \int_0^T e^{-\rho t} v(r_t) dt ,$$

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where ρ is a discount rate, $\rho \geq 0$, applied in continuous time to the utility flow $v(r_t)$ arising at any time t from a consumption flow r_t of the resource. The utility flow function $v(r)$ is defined for $r \geq \underline{r}$, is twice continuously differentiable and satisfies

$$(3a,b,c,d) \quad v'(r) > 0, \quad v''(r) < 0 \quad \text{for } r > \underline{r}, \quad v(\underline{r}) = 0,$$

$$\lim_{r \rightarrow \underline{r}} v'(r) = \infty .$$

That is, $v(r)$ is (a) strictly increasing and (b) strictly concave. The stipulation (c) anchors the utility scale. Some such anchoring, though not necessarily the given one, is needed whenever population size is a decision variable. The last requirement (d) simplifies a step in the proof, and can be secured if needed by a distortion of $v(r)$ in a neighborhood of \underline{r} that does not affect the solution.

The paper referred to gives an intuitive argument for the following

Theorem: For each $\rho \geq 0$ there exists a unique optimal path
 $r_t = \hat{r}_t, \quad 0 \leq t \leq \hat{T}_\rho$, maximizing (2) subject to

$$(4) \left\{ \begin{array}{l} (4a) \quad r_t \text{ is a continuous function on } [0, T] , \\ (4b) \quad \int_0^T r_t dt \leq R, \quad r_t \geq \underline{r}, \quad 0 \leq t \leq T . \end{array} \right.$$

For $\rho = 0$, the optimal path $(\hat{r}_t | 0 \leq t \leq \hat{T}_0)$ is defined by

$$(5) \begin{cases} (5a) & \hat{r}_t = \hat{r}, \text{ a constant, for } 0 \leq t \leq \hat{T}_0, \\ (5b) & v(\hat{r}) = \hat{r}v'(\hat{r}), \\ (5c) & \hat{r}\hat{T}_0 = R. \end{cases}$$

For $\rho > 0$ it is defined by

$$(6) \begin{cases} (6a) & e^{-\rho t} v'(\hat{r}_t) = e^{-\rho \hat{T}_\rho} v'(\hat{r}), \quad 0 \leq t \leq \hat{T}_\rho, \quad \hat{r} \text{ as in (5b)}, \\ (6b) & \int_0^{\hat{T}_\rho} \hat{r}_t dt = R. \end{cases}$$

The diagram illustrates the solution. For $\rho = 0$, (6) implies (5), and consumption of the resource is constant during survival. Its optimal level \hat{r} is obtained in (5b,c) by balancing the number of years of survival against the constant level of utility flow that the total resource stock makes possible during survival. Since $\hat{r} > \underline{r}$, the optimum survival period \hat{T}_0 is shorter than the maximum \bar{T} defined by (1).

For $\rho > 0$, the optimal path \hat{r}_t follows a declining curve given by (6a), which starts from a level \hat{r}_0 such that, when resource exhaustion brings life to a stop at time $t = \hat{T}_\rho$, the level $\hat{r}_{\hat{T}_\rho} = \hat{r}$ is just reached. Since the decline is steeper when ρ is larger, the survival period is shorter, the larger is ρ - which explains the title of this note.

The intuitive argument already referred to gives insight into the theorem; the following proof establishes its validity.

Proof: We first consider paths optimal under the added constraint of some arbitrarily fixed value $T = T^*$ of T satisfying $0 < T^* < \bar{T}$. Assume that such a " T^* - optimal" path r_t^* exists and that

$$(7) \quad r_t^* \geq \underline{r} + \delta \quad \text{for } 0 \leq t \leq T^* \text{ and some } \delta > 0 .$$

Then, if s_t is a continuous function defined for $0 \leq t \leq T^*$ such that

$$(8) \quad |s_t| \leq \delta \quad , \quad \int_0^{T^*} s_t dt = 0 \quad ,$$

the path

$$(9) \quad r_t = r_t^* + \epsilon s_t \quad , \quad 0 \leq t \leq T^* \quad ,$$

is T^* -feasible for $|\epsilon| \leq 1$ and satisfies

$$(10) \quad \left\{ \begin{array}{l} (10a) \\ (10b) \end{array} \right\} \begin{cases} V(\rho, T^*, (r_t)) - V(\rho, T^*, (r_t^*)) = \\ = \int_0^{T^*} e^{-\rho t} (v(r_t) - v(r_t^*)) dt = \\ = \epsilon \int_0^{T^*} e^{-\rho t} v'(r_t^*) s_t dt + R(\epsilon) \quad , \end{cases}$$

where the remainder $R(\epsilon)$ is of second order in ϵ . It is therefore a necessary condition for the T^* -optimality of r_t^* that

$$(11) \quad p_t \equiv e^{-\rho t} v'(r_t^*) = \text{constant} = e^{-\rho T^*} v'(r_{T^*}^*) \quad , \text{ say,}$$

because, if we had $p_{t'} \neq p_{t''}$, $0 \leq t', t'' \leq T^*$, we could by choosing s_t of one sign in a neighborhood in $[0, T^*]$ of t' , s_t of the opposite sign in one of t'' and zero elsewhere while preserving (8) make the last member of (10) positive for some ϵ with $|\epsilon| \leq 1$.

In the light of (3a,b), (11) justifies our assumption that r_t^* is a continuous function of t . We now find that r_t^* is constant for $\rho = 0$, strictly decreasing for $\rho > 0$. Given $r_{T^*}^*$, say, the solution r_t^* of (11) is uniquely determined, and, for each t , r_t^* is a strictly increasing differentiable function of the given $r_{T^*}^*$. Also, by (3d),

$$\lim_{r_{T^*}^* \rightarrow \underline{r}} \int_0^{T^*} r_t^* dt = \int_0^{T^*} \underline{r} dt = T^* \underline{r} < \overline{T} \underline{r} = R \quad ,$$

whereas, for sufficiently large $r_{T^*}^*$,

$$\int_0^{T^*} r_t^* dt > R \quad .$$

Therefore there is a unique number $\alpha^* > \underline{r}$ such that the unique solution r_t^* of (11) with $r_{T^*}^* = \alpha^*$ satisfies

$$(12) \quad \int_0^{T^*} r_t^* dt = R \quad .$$

From here on r_t^* will denote that path for the chosen T^* . Note that this path also satisfies (7).

To prove the unique T^* -optimality of r_t^* , let r_t be any T^* -feasible path such that $r_{t_0} \neq r_{t_0}^*$ for some $t_0 \in [0, T]$. Then, by the continuity of r_t , r_t^* , $r_t \neq r_t^*$ for all t in some neighborhood τ of t_0 in $[0, T^*]$. By (3b), for all $t \in [0, T^*]$,

$$(13) \quad v(r_t) - v(r_t^*) \begin{bmatrix} < \\ \leq \end{bmatrix} (r_t - r_t^*) v'(r_t^*) \quad \text{for } t \in \begin{bmatrix} \tau \\ T^* \end{bmatrix} \quad ,$$

where $\tau^* \equiv [0, T^*] - \tau$. Therefore, we have from (10a), (11), (4b) with $T = T^*$, and (12) that

$$\begin{aligned} & V(\rho, T^*, (r_t)) - V(\rho, T^*, (r_t^*)) = \\ & = \left(\int_{\tau} + \int_{\tau^*} \right) e^{-\rho t} (v(r_t) - v(r_t^*)) dt < \\ & < \int_0^{T^*} (r_t - r_t^*) e^{-\rho t} v'(r_t^*) dt = \\ & = e^{-\rho T^*} v'(r_{T^*}^*) \int_0^{T^*} (r_t - r_t^*) dt \leq 0 \quad . \end{aligned}$$

Hence r_t^* is uniquely T^* -optimal.

We now make T^* a variable, writing T instead of T^* and r_t^T instead of r_t^* . Note that, for each t , $0 \leq t < \bar{T}$, r_t^T is a differentiable function of T for $t \leq T < \bar{T}$. Therefore

$$V_T \equiv V(\rho, T, (r_t^T)) = \int_0^T e^{-\rho t} v(r_t^T) dt$$

is a differentiable function of T for $0 \leq T < \bar{T}$, and

$$\begin{aligned} \frac{dV_T}{dT} &= e^{-\rho T} v(r_T^T) + \int_0^T e^{-\rho t} v'(r_t^T) \frac{dr_t^T}{dT} dt = \\ &= e^{-\rho T} v(r_T^T) + e^{-\rho T} v'(r_T^T) \int_0^T \frac{dr_t^T}{dT} dt \end{aligned}$$

by (11). But, by (12),

$$0 = \frac{dR}{dT} = r_T^T + \int_0^T \frac{dr_t^T}{dT} dt .$$

Therefore,

$$e^{\rho T} \frac{dV_T}{dT} = v(r_T^T) - r_T^T v'(r_T^T) .$$

But then, from (5b), since $\frac{d}{dr} (v(r) - rv'(r)) = -rv''(r) > 0$ for $r > 0$, by (3b),

$$\frac{dV_T}{dT} \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0 \quad \text{for} \quad r_T^T \begin{bmatrix} < \\ = \\ > \end{bmatrix} \hat{r} .$$

Finally, since $0 < T < T' < \bar{T}$ implies $r_{T'}^{T'} \leq r_T^{T'} < r_T^T$,

$$\frac{dV_T}{dT} \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0 \quad \text{for} \quad T \begin{bmatrix} > \\ = \\ < \end{bmatrix} \hat{T}_\rho .$$

Thus, V_T reaches its unique maximum for that value \hat{T}_ρ of T for which $r_T^T = \hat{r}$.

This establishes the second part of the theorem. The first part follows by specialization when $\rho = 0$.

REFERENCE

Koopmans, T.C., "Some observations on 'optimal' economic growth and exhaustible resources", in Bos, Linnemann and de Wolff, Ed^s, Economic Structure and Development, essays in honour of Jan Tinbergen, Holland Publishing Co., 1973, pp. 239-55.

