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A JOB SHOP ASSIGNMENT PROBLEM
WITH QUEUING COSTS

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A Job Shop Assignment Problem with Queuing Costs

David E. Bell*

1. The Problem

Consider an assignment problem in which jobs are to be assigned to machines in such a way as to minimize the total cost of manufacture. In addition, there is, for each job, a queuing cost which is proportional to the time spent before completion. Each job takes a unit length of time to be completed once work is started on it by a machine.

It will be shown that this problem may be formulated as a linear program whose optimal solution will be integral.

For example, with four jobs ($i = 1, 2, 3, 4$) and two machines ($s = 1, 2$) with a fixed service cost of i times s plus a unit charge per period waited before completion, the optimal arrangement is to assign job 1 to machine 2 and the remainder to machine 1. This gives a total cost of

$$(1.2 + 1) + (2.1 + 3.1 + 4.1 + 1 + 2 + 3) = 18$$

2. The Formulation

Let r_i^s be the cost of processing job i on machine s . Let $x_{is} = 1$ if job i is assigned to machine s and 0 otherwise. Let $y_{ks} = 1$ if machine s has k jobs assigned to it and 0 otherwise.

*Carlos Winkler supplied the neat proof of the theorem. This problem was suggested by Aleksandr Butrimenko.

The following integer program models the situation

$$\begin{aligned}
 \min \quad & \sum_{i,s} r_i^s x_{is} + \sum_{s,k} \frac{k(k+1)}{2} y_{ks} \\
 & \sum_s x_{is} = 1 \quad \text{for each } i \\
 & \sum_k y_{ks} = 1 \quad \text{for each } s \\
 & \sum_k k y_{ks} = \sum_i x_{is} \quad \text{for each } s \quad . \\
 & y_{ks} \geq 0 \quad x_{is} \geq 0 \\
 & x_{is} \text{ integer}
 \end{aligned}$$

Note that it is not necessary to enforce the integrality of the y variables as y_{ks} will be integral if $\sum_i x_{is}$ is integral, because of the form of the objective function. Note too*, that if the y 's are integral in the optimal solution, then so will the x 's be integral because for fixed integral y 's, the problem is just an assignment problem, which is known to solve in integers.

Lemma In the optimal solution to the problem

$$\sum_i x_{is} \text{ integer} \Rightarrow y_{ks} \text{ integer for all } k$$

and all y_{ks} integer \Rightarrow whole solution is integral.

Theorem The optimal solution to the linear program (assumed to be an extreme point) is integral.

*Observation by George Dantzig

Proof The lemma only leaves the case where at least one $\sum_i x_{is}$ is not integral. It will be shown that such an optimal solution is not extreme. Suppose that

$$0 < x_{is}^* < 1$$

in the optimal solution. Hence there exists some j for which $0 < x_{ij}^* < 1$ for the same i . Suppose first that $\sum_p x_{pj}^*$ is not integral. Then we may find an $\epsilon > 0$ such that

$$k_1 < \sum_p x_{ps}^* \pm \epsilon < k_1 + 1$$

$$k_2 < \sum_p x_{pj}^* \pm \epsilon < k_2 + 1$$

Associated with the three solutions (x_{ij}, x_{is}) , $(x_{ij}^* + \epsilon, x_{is}^* - \epsilon)$ and $(x_{ij}^* - \epsilon, x_{is}^* + \epsilon)$ are the solutions $(y_{k_1s}^*, y_{k_1+1,s}^*, y_{k_2j}^*, y_{k_2+1,j}^*)$, $(y_{k_1s}^* + \epsilon, y_{k_1+1,s}^* - \epsilon, y_{k_2j}^* - \epsilon, y_{k_2+1,j}^* + \epsilon)$ and $(y_{k_1s}^* - \epsilon, y_{k_1+1,s}^* + \epsilon, y_{k_2j}^* + \epsilon, y_{k_2+1,j}^* - \epsilon)$.

The important point is that only these variables are affected. All three solutions are feasible and the optimal solution is a linear combination of the other two. Hence, the optimal solution is not extreme. Now the case when $\sum_p x_{pj}^*$ is integral must be considered. In this case, since x_{ij}^* is not integral, x_{aj}^* must be non integral for some $a \neq i$. Hence, x_{at}^* is not integral for some

$t \neq j$. If $t = s$, then the solution

$$(x_{is}^* + \epsilon, x_{ij}^* - \epsilon, x_{aj}^* + \epsilon, x_{as}^* - \epsilon)$$

is feasible without affecting the y 's. The same argument about non-extremeness then applies. If $\sum_p x_{pt}^*$ is integral, the system is repeated. If it is not integral, then the first argument still applies. In summary, the argument is just that of the assignment problem proof, except that the y variables may be affected. Since these respond linearly to changes in the x variables, all is well. //

As empirical evidence of the truth of the theorem, two problems having 21 jobs and 6 machines solved in integers.