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A GROUP UTILITY FUNCTION BY OPTIMAL  
ALLOCATION

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Consider an imaginary research institute where the director has to make decisions under uncertainty concerning the institute's research funds which when received, must be allocated by him between several projects. The project leaders have supplied him with their utility functions  $u_1, u_2, \dots, u_n$  for their own project funding, that is, project leader  $k$  uses utility function  $u_k$  in matters concerning funding for his own project.

Now the director decides that, given an amount of research funding  $x$ , he will allocate an amount  $x_k$  to project  $k$  so as to maximize

$$u_1(x_1) + u_2(x_2) + \dots + u_n(x_n)$$

subject to the constraint  $x_1 + \dots + x_n = x$ .

The question considered here, posed to me by Ralph L. Keeney, is, what is the implied utility function of the director?.

Note that what follows applies also to the case where the director chooses to maximize

$$k_1 u_1(x_1) + \dots + k_n u_n(x_n)$$

for some positive weights  $k_i$  since utility function  $u_i$  may be rescaled without loss to  $k_i u_i$ .

Let

$$u(x) = \text{Max } [u_1(x_1) + \dots + u_n(x_n)]$$

$$\text{s.t. } x_1 + \dots + x_n = x$$

Result 1 It is sufficient to consider the case  $n = 2$  since a general case may be obtained by repeated application of the  $n = 2$  case.

Proof.

We require

$$u(x) = \max_{x_1+x_2+x_3 = x} [u_1(x_1) + u_2(x_2) + u_3(x_3)]$$

but let us consider first the function

$$v(y) = \max_{y_2+y_3 = y} [u_2(y_2) + u_3(y_3)] .$$

It will be sufficient for the result to show that

$$u(x) = \max_{x_1+y = x} [u_1(x_1) + v(y)] .$$

But

$$\begin{aligned} & \max_{x_1+y = x} [u_1(x_1) + \max_{y_2+y_3 = y} (u_2(y_2) + u_3(y_3))] . \\ &= \max_{x_1+y_2+y_3 = x} [u_1(x_1) + u_2(y_2) + u_3(y_3)] = u(x) . || \end{aligned}$$

Hence until further notice this paper will consider the problem

$$u(x) = \max_y u_1(y) + u_2(x - y) . \quad (1)$$

For clarification of notation

$$f'(x) = \left. \frac{d}{dt} f(t) \right|_{t=x}$$

so, for example,

$$\frac{d}{dx}u(y-x) = u'(y-x)\left(\frac{dy}{dx}-1\right)$$

where

$$u'(y-x) = \left. \frac{du(t)}{dt} \right|_{t=y-x}$$

Result 2 An optimal  $y$  exists and is unique for (1) if  $u_1$  and  $u_2$  are strictly concave.

Proof First note that  $u_1(y) + u_2(x - y)$  is strictly concave in  $y$  since  $u_1$  and  $u_2$  are, and the sum of two concave functions is concave.

Hence a maximum is attained and is unique. ||

Define  $y(x)$  by the relation

$$u(x) = u_1(y(x)) + u_2(x - y(x))$$

then result 2 shows that  $y(x)$  is well defined on  $x$ .

Result 3 If  $u_1$  and  $u_2$  are twice differentiable and strictly concave then

$$u'(x) = u'_1(y(x)) = u'_2(x - y(x)) \quad .$$

Proof

Let

$$\theta(y) = u_1(y) + u_2(x - y)$$

then

$$\frac{d\theta}{dy} = u'_1(y) - u'_2(x - y) \quad .$$

Hence

$$\frac{d\theta}{dy} = 0 \quad \text{where} \quad u'_1(y) = u'_2(x - y) \quad . \quad (2)$$

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This is a maximum since  $\theta(y)$  is strictly concave in  $y$ . Equation (2) will always be satisfied if it is assumed that the ranges of  $u_1$  and  $u_2$  are equal. For most utility functions this range is  $(-\infty, 0)$ , and thus (2) will hold.

Using a Taylor expansion

$$u(x) + \delta x u'(x) = u_1(y) + \delta y u_1'(y) + u_2(x-y) + (\delta x - \delta y) u_2'(x-y). \quad (3)$$

and (1) and (2) give that

$$\delta x u'(x) = \delta y (u_1'(y)) + (\delta x - \delta y) u_2'(x-y) = \delta x u_1'(y) \quad ||$$

Result 4 If the total funding  $x$  is increased then each group receives an increased allocation.

Proof Using a Taylor Expansion for the optimal  $y = y(x)$ ,

$$u(x + \delta x) = u_1(y + \delta y) + u_2(x + y + \delta x - \delta y)$$

so that

$$\begin{aligned} u(x + \delta x) = & u_1(y) + \delta y u_1'(y) + \frac{\delta y^2}{2} u_1''(y) \\ & + u_2(x-y) + (\delta x - \delta y) u_2'(x-y) + \frac{(\delta x - \delta y)^2}{2} u_2''(x-y) \end{aligned}$$

using (1) and (2) gives

$$u(x + \delta x) = u(x) + \delta x u'(x) + \frac{\delta y^2}{2} u_1''(y) + \frac{(\delta x - \delta y)^2}{2} u_2''(x-y). \quad (4)$$

Maximizing (4) with respect to  $\delta y$  implies that

$$\delta y u_1''(y) - (\delta x - \delta y) u_2''(x - y) = 0$$

that is

$$\frac{\delta y}{\delta x} = \frac{u_2''(x-y)}{u_1''(y) + u_2''(x-y)} > 0$$

since  $u_1$  and  $u_2$  are strictly concave.

Also

$$1 - \frac{\delta y}{\delta x} = \frac{u_1''(y)}{u_1''(y) + u_2''(x-y)} > 0$$

also so that each group receives a strict increase in allocation. ||

Theorem 1 Assuming

(i)  $u_1$  and  $u_2$  are strictly increasing, strictly concave with continuous second derivatives

(ii)  $\left. \begin{array}{ll} u_i'(x) \rightarrow 0 & \text{as } x \rightarrow \infty \\ \rightarrow -\infty & \text{as } x \rightarrow -\infty \end{array} \right\} \quad i = 1, 2,$

then

$$u(x) = (u_1 + u_2 u_2'^{-1} u_1') [(1 + u_2'^{-1} u_1')^{-1}(x)] .$$

Proof We have from (2) that

$$u_1'(y) = u_2'(x-y)$$

and since  $u_2'$  is strictly decreasing and continuous it has an inverse  $u_2'^{-1}$  which is valid over the range of  $u_1'$ , so that

$$x - y = u_2'^{-1} u_1'(y) \tag{5}$$

or

$$x = y + u_2'^{-1} u_1'(y) = (1 + u_2'^{-1} u_1') (y) .$$

Since  $\frac{dy}{dx} > 0$  and  $u_2^{-1}$  is continuous we have that

$1 + u_2^{-1} u_1$ , is strictly increasing and continuous so that it too has an inverse, hence

$$y = (1 + u_2^{-1} u_1)^{-1} (x) \quad (6)$$

Since

$$u(x) = u_1(y) + u_2(x - y)$$

from (5) and (6) we have that

$$\begin{aligned} u(x) &= (u_1 + u_2 u_2^{-1} u_1) (y) \\ &= (u_1 + u_2 u_2^{-1} u_1) [(1 + u_2^{-1} u_1)^{-1} (x)] \quad || \\ &\quad (7) \end{aligned}$$

Note that the formula is entirely in terms of the known functions  $u_1$  and  $u_2$ .

#### Examples

Take  $u_1(x) = \log(x + a)$   $u_2(x) = \log(x + b)$   
so the problem is, find  $u(x)$  such that

$$u(x) = \max_y [\log(y + a) + \log(x - y + b)]$$

By differentiating, setting to to zero we have

$$\frac{1}{y + a} - \frac{1}{x - y + b} = 0$$

$$\therefore x - y + b = y + a$$

$$\therefore y = \frac{x - a + b}{2}$$

Hence

$$\begin{aligned} u(x) &= \log\left(\frac{x - a + b}{2} + a\right) + \log\left(\frac{x - b + a}{2} + b\right) \\ &= \underline{\underline{2 \log\left(\frac{x + a + b}{2}\right)}} \end{aligned}$$

Now to demonstrate that the same result may be had using (7).

$$u_1'(x) = \frac{1}{x + a} \quad u_2'(x) = \frac{1}{x + b}$$

Thus

$$\begin{aligned} u_2^{-1}(x) &= \frac{1}{x} - b \\ \therefore y(x) &= (1 + u_2^{-1} u_1')(x) = x + \frac{1}{u_1'(x)} - b \\ &= 2x + a - b \end{aligned}$$

$$\therefore (1 + u_2^{-1} u_1')^{-1}(x) = \frac{x - a + b}{2}.$$

Now

$$(u_1 + u_2 u_2^{-1} u_1')(x) = \log(x + a) + \log(x + a - b + b)$$

Hence

$$\begin{aligned} (u_1 + u_2 u_2^{-1} u_1') [(1 + u_2^{-1} u_1')^{-1}(x)] \\ &= 2 \log\left(\frac{x - a + b}{2} + a\right) \\ &= 2 \log\left(\frac{x + a + b}{2}\right) \text{ as required.} \end{aligned}$$


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Hence  $u_1 = \log(x + a)$   $u_2 = \log(x + b)$  yields a utility function of  $\log(x + a + b)$  for the director.

As an exercise for the reader, it may be shown either by direct route or via (7) that if

$$u_1(x) = -e^{-cx} \quad \text{and} \quad u_2(x) = -e^{-dx}$$

then

$$u(x) = -e^{-\frac{cd}{c+d}x} \left( \left(\frac{c}{d}\right)^{-\frac{c}{c+d}} + \left(\frac{d}{c}\right)^{-\frac{d}{c+d}} \right)$$

Hence  $u_1(x) = -e^{-cx}$  ,  $u_2(x) = -e^{-dx}$  yields a utility function of  $-e^{-\frac{cd}{c+d}x}$  for the director.

In these two standard cases  $u(x)$  is the same form as the functions  $u_1$  and  $u_2$ .

Indeed if

$$u_1(x) = -me^{-cx} \quad u_2(x) = -ne^{-dx}$$

then

$$u(x) \sim -e^{-\frac{cd}{c+d}x}$$

that is, the directors weighting of the projects does not affect his decisions.

#### Theorem 2

If each member of the group has

(i) a utility function  $-e^{-c_k x}$   $k = 1, \dots, n$

(ii) a utility function  $\log(x + a_k)$   $k = 1, \dots, n$

or (iii) a utility function  $-(x + b_k)^{-p}$   $p > 0$ ,  $k = 1, \dots, n$

then the group utility function  $u$  is independent of the weightings given to the individual members utilities.

Proof Since it is merely a matter of solution the proof will be omitted. However the statement of the theorem will be amplified.

(i) If

$$u(x) = \max w_1 u_1 + w_2 u_2 + \dots + w_n u_n$$

where

$$u_k(x) = -e^{-c_k x}$$

then

$$u(x) = -f(w_1, \dots, w_n, c_1, \dots, c_n) e^{-cx}$$

where

$$f(\cdot) > 0 \quad \text{and}$$

$$\frac{1}{c} = \frac{1}{c_1} + \dots + \frac{1}{c_n}.$$

(ii) If

$$u_k(x) = \log(x + a_k)$$

then

$$u(x) = g(a_1, \dots, a_n, w_1, \dots, w_n) \log(x + a)$$

where

$$a = a_1 + \dots + a_n \quad g(\cdot) > 0.$$

(iii) If

$$u_k(x) = -(x + b_k)^{-p} \text{ for some constant } p > 0$$

then

$$u(x) = -h(w_1, \dots, w_n, p) (x + b)^{-p} \quad h(\cdot) > 0.$$

where  $b = b_1 + \dots + b_n$ . ||

The importance of this result lies in the observation that the group decision maker need only know the group

members' utilities before making decisions on group funds. He need not decide beforehand how he will weight the importance of the group members. In particular the role of group decision maker and weighting decision maker may be divided between two different people (or groups) who need not even communicate with each other.

The interestingly simple expression for the exponential coefficient in Theorem 2 (i) can be generalized.

Theorem 3 If  $r(x)$  is the coefficient of risk for the group utility function and  $r_k(x)$  that for the individual members then

$$\frac{1}{r(x)} = \frac{1}{r_1(x_1)} + \frac{1}{r_2(x_2)} + \dots + \frac{1}{r_n(x_n)}$$

where  $x_k$  is the optimal allocation to group member  $k$ .

Proof Recall that  $r(x) = -\frac{u''(x)}{u'(x)}$ .

Note that this result is true whatever the forms of the  $u_k$  but that this does not imply the proof of a general form of Theorem 2 because the optimal  $x_k$  will depend on the weightings.

For simplicity's sake we will prove it for the case  $n = 2$ .

Now

$$u'(x) = u'_1(y) = u'_2(x - y) \quad (8)$$

for an optimal  $y$ .

Thus, differentiating with respect to  $x$ ,

$$u''(x) = u_1''(y) \frac{dy}{dx}$$

From the proof of Result 4

$$\begin{aligned} \frac{dy}{dx} &= \frac{u_2''(x-y)}{u_1''(y) + u_2''(x-y)} \\ \therefore u''(x) &= \frac{u_1''(y) u_2''(x-y)}{u_1''(y) + u_2''(x-y)} \end{aligned}$$

that is

$$\frac{1}{u''(x)} = \frac{1}{u_1''(y)} + \frac{1}{u_2''(x-y)}$$

Using (8) once more we have that

$$-\frac{u'(x)}{u''(x)} = -\frac{u_1'(y)}{u_1''(y)} - \frac{u_2'(x-y)}{u_2''(x-y)}$$

which gives the result. The extension to a general  $n$  is trivial using Result 1. ||

The following is a characterization (perhaps not complete) of functions having the property of Theorem 2.

Theorem 4 If all members of the group have a utility function

$$u_k(x) = v(a_k x + b_k)$$

then  $u(x)$  the group utility function, is independent of the weightings assigned to the group members if

$$(i) \quad \theta v(x) = v(x + f(\theta)) \quad (\text{where } f^{-1} \text{ exists})$$

or if  $(ii) \quad v(\theta x) = v(x) + g(\theta)$  .

Proof Note that it is sufficient for (i) to take  $b_k = 0$  and for (ii) to take  $a_k = 1$ .

We will also just prove it for the case  $n = 2$ .

(i) At optimality

$$a_1 \theta v'(a_1 y) = a_1 v'(a_2(x - y))$$

or using property (i)

$$v'(a_1 y + f(\theta)) = v'(a_2(x - y))$$

Since  $v'^{-1}$  exists

$$a_1 y + f(\theta) = a_2(x - y)$$

and

$$y = \frac{a_2 x - f(\theta)}{a_1 + a_2}$$

$$\begin{aligned}
 u(x) &= \theta v\left(\frac{a_1 a_2 x - a_1 f(\theta)}{a_1 + a_2}\right) + v\left(\frac{a_1 a_2 x + a_2 f(\theta)}{a_1 + a_2}\right) \\
 &= v\left(\frac{a_1 a_2 x - a_1 f(\theta)}{a_1 + a_2} + f(\theta)\right) + v\left(\frac{a_1 a_2 x}{a_1 + a_2} + \frac{a_2 f(\theta)}{a_1 + a_2}\right) \\
 &= 2v\left(\frac{a_1 a_2 x}{a_1 + a_2} + \frac{a_2 f(\theta)}{a_1 + a_2}\right) \\
 &= 2f^{-1}\left(\frac{a_2 f(\theta)}{a_1 + a_2}\right) \left(v\left(\frac{a_1 a_2 x}{a_1 + a_2}\right)\right)
 \end{aligned}$$

which is independent of the weighting  $\theta$ .

(ii) At optimality

$$v(y + b_1) = v(b_2 + x - y)$$

Now  $\theta v(x) = v\left(\frac{x}{\theta}\right)$  from (ii)

hence

$$\frac{y + b_1}{\theta} = b_2 + x - y$$

or  $y = \frac{\theta(b_2 + x) - b_1}{1 + \theta}$

$$\begin{aligned}
 u(x) &= \theta v\left(\frac{\theta(b_1 + b_2 + x)}{1 + \theta}\right) + v\left(\frac{b_1 + b_2 + x}{1 + \theta}\right) \\
 &= (1 + \theta) v(b_1 + b_2 + x) + g\left(\frac{\theta}{1 + \theta}\right) + g\left(\frac{1}{1 + \theta}\right)
 \end{aligned}$$

which again is independent of the weightings.

||

Concluding Remarks Results have been presented when a group decision maker allocates resources using the criterion of maximizing total utilities. This criterion has not been justified, although Theorem 2 suggests to me that it may well be reasonable and certainly simple to use.

Additivity is not essential for many of the results. For example if the criterion used is

$$u(x) = \max_{x_1 + \dots + x_n = x} u_1(x_1) u_2(x_2) \dots u_n(x_n) .$$

Many of the results are actually simpler because the problem of weightings does not arise. The results of this paper may be applied very straightforwardly to the situation

$$\log u(x) = \max_{x_1 + \dots + x_n = x} \log u_1(x_1) + \dots + \log u_n(x_n) .$$