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MULTIREGIONAL AGE-STRUCTURED POPULATIONS WITH CHANGING RATES: WEAK AND STOCHASTIC ERGODIC THEOREMS

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1. INTRODUCTION

A biological population, human or nonhuman, may experience multiple states in two ways.

First, it may visit different states in the course of time, the whole population experiencing the same (possibly age-specific) vital rates at any one time. For example, a troop of baboons moves from one area to another of its range, with associated changes in food supply and risks of predation (Altmann and Altmann, 1970). A human population experiences fluctuating crop yields from one year to the next, with associated effects on childbearing and survival. There are *serial* changes of state of a *homogeneous* population.

Second, the population may be subdivided into *inhomogeneous* subpopulations that exercise different states in *parallel*. Individuals may migrate from one state to another in the course of time. The states may correspond to geographical regions, work status, marital status, health status, or other classifications (Rogers, 1980).

The purpose of this paper is to describe population models in which serial and parallel inhomogeneity are combined. In demography, theorems that describe long-run behavior that is independent of initial conditions are called ergodic theorems. Weak ergodic theorems assume that the rates that govern a population's evolution themselves follow some deterministic trajectory. Le Bras (1977) gave the first weak ergodic theorem for multiregional age-structured populations. We shall give four weak ergodic theorems that are more general than that of Le Bras. Stochastic ergodic theorems assume that the rates that govern a population's evolution are selected from a set of possible rates by some stochastic process. We shall state a stochastic ergodic theorem that assumes that the rates of birth, death, and migration or other transition are selected by a Markov chain.

2. THE FORMALISM OF MULTIREGIONAL POPULATION MODELS

Following Rogers (1966), we now describe a formalism commonly used for projecting multiregional age-structured populations. Though we speak of regions and of migration, appropriate terminology for other states could be substituted.

Let r be the number of regions and k be the number of age classes. A census by age and region $Y(t)$ is an r k-vector partitioned into r k-vectors $Y_j(t,.)$, $j = 1,\ldots,r$, in which the ith element $Y_j(t,i)$, $i = 1,\ldots,k$, is the number of individuals at time t in region j in age class i. X is a set of $(rk) \times (rk)$ nonnegative matrices. A typical matrix $\underset{\sim}{x}$ in X is partitioned into r^2 $k \times k$ submatrices $\underset{\sim}{x}_{gh}$, g, $h = 1,\ldots,r$, one such submatrix for each ordered pair (g,h) of regions.

Censuses are assumed to evolve according to the recursion

$$Y(t+1) = \underset{\sim}{x}(t+1)Y(t), \qquad t = 0,1,2,\ldots \tag{2.1}$$

where $\underset{\sim}{x}(t+1)$ is a matrix chosen from X. If $\underset{\sim}{x}(t+1) = \underset{\sim}{x}$, then the element $\underset{\sim}{x}_{gh}(1,j)$ of the submatrix $\underset{\sim}{x}_{gh}$ is the average number of individuals born from t to $t+1$, per individual in region h and age class j at time t, who are alive in region g at $t+1$; g, $h = 1,\ldots,r$; $j = 1,\ldots,k$. Also $\underset{\sim}{x}_{gh}(j+1,j)$ is the proportion of individuals in age class j and region h at time t who are alive in age class $j+1$ and region g at time $t+1$; $j = 1,\ldots,k-1$. The remaining elements of $\underset{\sim}{x}_{gh}$ are zero.

In the case of 2 regions, r = 2, and 2 age classes, k = 2, $\underset{\sim}{x}$ and Y(t) have the form

$$\underset{\sim}{x} = \left(\begin{array}{cc|cc} \underset{\sim}{x}_{11}(1,1) & \underset{\sim}{x}_{11}(1,2) & \underset{\sim}{x}_{12}(1,1) & \underset{\sim}{x}_{12}(1,2) \\ \underset{\sim}{x}_{11}(2,1) & 0 & \underset{\sim}{x}_{12}(2,1) & 0 \\ \hline \underset{\sim}{x}_{21}(1,1) & \underset{\sim}{x}_{21}(1,2) & \underset{\sim}{x}_{22}(1,1) & \underset{\sim}{x}_{22}(1,2) \\ \underset{\sim}{x}_{21}(2,1) & 0 & \underset{\sim}{x}_{22}(2,1) & 0 \end{array}\right), \qquad Y(t) = \left(\begin{array}{c} Y_1(t,1) \\ Y_1(t,2) \\ \hline Y_2(t,1) \\ Y_2(t,2) \end{array}\right)$$

There is a 2×2 submatrix of $\underset{\sim}{x}$ for each region and the elements within each submatrix refer to age classes. An alternate arrangement of elements by age class is described, e.g., by Willekens and Rogers (1978), following Feeney (1970).

3. WEAK ERGODIC THEOREMS FOR MULTIREGIONAL POPULATION MODELS

We now introduce some concepts needed to state ergodic theorems for multiregional (or multistate) populations.

For any nonnegative vectors u and v of the same length, with elements u_i and v_i respectively, define the Hilbert projective pseudometric d(u,v) by

$$d(u,v) = \log[\max_{v_i>0}(u_i/v_i)/\min_{v_j>0}(u_j/v_j)]$$

if u and v have positive elements in corresponding positions; and by

$$d(u,v) = \infty \text{ if } (u_i = 0 \text{ and } v_i > 0) \text{ or } (u_i > 0 \text{ and } v_i = 0)$$

for some i. Here d measures how nearly the elements of u are proportional to the corresponding elements of v; d(u,v) = 0 if and only if u = cv for some scalar $c > 0$. Thus if $Y_1(t)$ and $Y_2(t)$, t = 0,1,2,... are two sequences of age censuses, then as $t \to \infty$, $d(Y_1(t), Y_2(t)) \to 0$ if and only if the corresponding distributions of the population by age and region eventually differ by a vanishingly small amount.

We now define four kinds of sets of nonnegative matrices and discuss the relations among them: a contracting set, an exponentially contracting set, a primitive set, and an ergodic set.

A contracting set S (as defined in Cohen, 1979, p. 354) is a set of $n \times n$ matrices ($1 \leq n \leq \infty$) such that if u and v are any two positive n-vectors, then for any $\varepsilon > 0$ there is an integer N (possibly depending on u and v) such that for all $q \geq N$, and for any sequence $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_q, \ldots$ of matrices chosen from S, if $H(1,q) = \underset{\sim}{x}_q \ldots \underset{\sim}{x}_1$, then $d(H(1,q)u, H(1,q)v) < \varepsilon$.

A contracting set S is an exponentially contracting set (Cohen, 1979, p. 354) if, for any positive n-vectors u and v, there exist positive constants $K < 1$ and D (with D possibly depending on u and v) such that for any products $H(1,t)$ of t arbitrary matrices from S, $d(H(1,t)u, H(1,t)v) \leq DK^t$.

A primitive set S with parameters (n,q), where n and q are positive integers, is a set of $n \times n$ nonnegative matrices such that any product of q factors which are matrices in S is positive (i.e. every element of the product is positive). An ergodic set S (Hajnal, 1976) with parameters (n,q,R) where $R > 0$ is a primitive set with parameters (n,q) such that for any matrix $\underset{\sim}{m} \in S$, $\min^+(\underset{\sim}{m})/\max(\underset{\sim}{m}) > R > 0$. Here $\min^+(\underset{\sim}{m})$ and $\max(\underset{\sim}{m})$ are the smallest and largest of the positive elements of $\underset{\sim}{m}$.

Every matrix $\underset{\sim}{m}$ in an ergodic set must be primitive, that is, have some power that is positive. But not every collection of primitive matrices is an ergodic set. For example, if

$$\underset{\sim}{m}_1 = \begin{pmatrix} + & 0 & + \\ + & 0 & 0 \\ 0 & + & 0 \end{pmatrix}, \quad \underset{\sim}{m}_2 = \begin{pmatrix} 0 & + & + \\ + & 0 & 0 \\ 0 & + & 0 \end{pmatrix},$$

where the + sign indicates some positive number, both $\underset{\sim}{m}_1$ and $\underset{\sim}{m}_2$ are primitive matrices in Leslie form. But

$$\underset{\sim}{m}_1^2 \underset{\sim}{m}_2 = \begin{pmatrix} + & + & + \\ 0 & + & + \\ 0 & + & + \end{pmatrix}$$

is reducible, so that no matrix of the form $(\underset{\sim}{m}_1^2\underset{\sim}{m}_2)^n$, $n \geq 0$, is positive or even irreducible. Hence $\{\underset{\sim}{m}_1, \underset{\sim}{m}_2\}$ is not an ergodic set.

An ergodic set is an exponentially contracting set (Hajnal, 1976). But ergodic sets have a uniform property not necessarily enjoyed by exponentially contracting sets in general. If S is an ergodic set with parameters (n,q,R), then there exist constants $D > 0$ and $K \in (0,1)$ such that for all initial n-vectors u, v, if $u \geq 0$, $v \geq 0$, $u \neq 0$, $v \neq 0$, then $d(H(1,t)u, H(1,t)v) \leq DK^t$, as soon as $t \geq q$. The point here is that D does not depend on the initial u and v. This fact is stated by Golubitsky, Keeler and Rothschild (1975, p. 89) for products of matrices $\underset{\sim}{x}$ satisfying $\underset{\sim}{A} \leq \underset{\sim}{x} \leq \underset{\sim}{B}$, where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are fixed primitive matrices. Their argument carries over immediately to ergodic sets. In particular, even if $d(u,v) = \infty$, $\underset{\sim}{H}^* = H(1,q) > 0$ implies $d(\underset{\sim}{H}^*u, \underset{\sim}{H}^*v) \leq \sup_{H^*} \max_{g,h,i,j} H^*_{gh}H^*_{ij}/(H^*_{ih}H^*_{gj}) \leq (n/R)^{2q} < \infty$, where the supremum is taken over every possible product $\underset{\sim}{H}^*$ of q arbitrary matrices from S.

Exponentially contracting sets need not display such uniformity. For example, the set S containing only the matrix

$$\underset{\sim}{x} = \begin{pmatrix} 1 & 0 \\ 1 & 1/2 \end{pmatrix}$$

is an exponentially contracting, but not ergodic, set. Let $u^T = (0,1)$, $v^T = (1,1)$. Then $\underset{\sim}{x}^t u = (0, 2^{-t})^T$ while $\underset{\sim}{x}^t v > 0$ for all t, so $d(\underset{\sim}{x}^t u, \underset{\sim}{x}^t v) = \infty$ for all t. If $u(\varepsilon) = (\varepsilon, 1)^T$, then $\underset{\sim}{x}^t u(\varepsilon) = (\varepsilon, \delta + 2^{-t})$, where δ can be made arbitrarily small by taking ε small. Consequently, for any fixed t, $d(\underset{\sim}{x}^t u(\varepsilon), \underset{\sim}{x}^t v)$ can be made arbitrarily large by making ε small enough.

An obvious way to assure that D in the upper bound DK^t is independent of the initial vectors u, v is to take initial vectors only from the set $Y(\delta) = \{y > 0 \,;\, \min_i y_i / \max_j y_j \geq \delta\}$. In this case, D depends on δ, not on $u, v \in Y(\delta)$.

Of the four kinds of sets just defined, only ergodic sets and exponentially contracting sets will appear in the following theorems. We still need two more concepts, that of an incidence matrix and that of a state connection matrix.

The incidence matrix $k(\underset{\sim}{A})$ of any matrix $\underset{\sim}{A} = (a_{ij})$ is the matrix whose elements $k_{ij}(\underset{\sim}{A})$ satisfy $k_{ij}(\underset{\sim}{A}) = 1$ if $a_{ij} \neq 0$ and $k_{ij}(\underset{\sim}{A}) = 0$ if $a_{ij} = 0$. Hajnal (1976) observed that if S is a set of square nonnegative matrices $\underset{\sim}{s}$ all of which have a common incidence matrix $\underset{\sim}{k}$ which is primitive, and if $\min^+(\underset{\sim}{s})/\max(\underset{\sim}{s}) > R > 0$ for all $\underset{\sim}{s}$ in S, then S is an ergodic set.

The state connection matrix is a generalization of the incidence matrix.

If $\underset{\sim}{x}$ is a $kr \times kr$ multistate projection matrix, as described earlier, define the state connection matrix $c(\underset{\sim}{x})$ to be the $r \times r$ matrix with $c_{gh}(\underset{\sim}{x}) = 0$ if every element of the $k \times k$ submatrix $\underset{\sim}{x}_{gh}$ of $\underset{\sim}{x}$ is zero, and $c_{gh}(\underset{\sim}{x}) = 1$ if there is at least one positive element in $\underset{\sim}{x}_{gh}$.

We can now state a weak ergodic theorem.

THEOREM 3.1. Let X be a set of multiregional projection matrices for r states with k age classes. Suppose that

(i) for every $\underset{\sim}{x} \in X$, $\min^+(\underset{\sim}{x})/\max(\underset{\sim}{x}) > R > 0$;

(ii) all matrices $\underset{\sim}{x}$ in X have the same incidence matrix κ;

(iii) each diagonal $k \times k$ submatrix κ_{gg} of κ is primitive, $g = 1,\ldots,r$ and $c(\kappa)$ is irreducible.

Then X is an ergodic set with parameters (kr,q,R), where $q = (r-1)(2k^2-4k+5)$.

Theorem 3.1 goes beyond the weak ergodic theorem of Le Bras (1977). Our assumption (i) is a bound only on the ratios of the positive elements within one matrix. Over the set X matrix elements may be arbitrarily large or small. Le Bras, like Golubitsky, Keeler and Rothschild (1975), assumes fixed upper and lower bounds on the elements of the multistate projection matrices.

Proof of Theorem 3.1. Correcting an assertion of Le Bras (1971), Feeney (1971) proved that if every diagonal submatrix $\underset{\sim}{x}_{gg}$, $g = 1,\ldots,r$, of a multistate projection matrix is primitive and if $c(\underset{\sim}{x})$ is irreducible, then $\underset{\sim}{x}$ is primitive. Therefore, by (iii), the common incidence matrix of every matrix in X is primitive, hence X is an ergodic and exponentially contracting set (Hajnal, 1976).

To derive q, we note that if $\underset{\sim}{A}$ is $n \times n$ and primitive, then $\underset{\sim}{A}^p > 0$ for $p \leq n^2 - 2n+2$ (Berman and Plemmons, 1979, p. 48). Take $p = k^2 - 2k+2$. If $H(1,p)$ is a product of p arbitrary matrices chosen from X, then the diagonal $k \times k$ submatrices of $H(1,p)$ are positive. Therefore the g,h submatrix of $H(1,p+1)$ has a strictly positive column if $c_{gh}(\kappa) = 1$, so $H(1,p+1+p) = H(1,2p+1)$ has a positive g,h submatrix if $c_{gh}(\kappa) = 1$. Now since $c_{gg}(\kappa) = 1$, $g = 1,\ldots,r$, $[c(\kappa)]^{r-1} > 0$ (Berman and Plemmons, 1979, p. 27). Since the product of any two positive (sub)matrices of the same size is positive, and since positive elements of $c(\kappa)$ correspond to positive submatrices of $H(1,2p+1)$, we see that every element of $H(1,(r-1)(2p+1))$ is positive. Thus $q = (r-1)(2k^2 - 4k+5)$, as asserted. This proves Theorem 3.1.

Since κ is primitive, by the result of Feeney (1971), we could have immediately written $q = (kr)^2 - 2kr+2$. However, it is easy to show that if $r > 1$ and $k > 1$, then $(kr)^2 - 2kr+2 > (r-1)(2k^2 - 4k+5)$, so the value of q stated in the theorem is preferable. For $r = 4$, $k = 10$, $(kr)^2 - 2kr+2 = 1522$ whereas $(r-1)(2k^2 - 4k+5) = 495$.

We now weaken conditions (ii) and (iii) of Theorem 3.1.

THEOREM 3.2. Let X be a set of multistate projection matrices for r states with k age classes. Suppose, in addition to (i) of Theorem 3.1, that (ii) for each $g = 1,\ldots,r$, $\{\underset{\sim}{x}_{gg}; \underset{\sim}{x} \in X\}$ is a primitive set with parameters (k,q_g); (iii) $\{c(\underset{\sim}{x}); \underset{\sim}{x} \in X\}$ is a primitive set with parameters (r,q_0). Then X is an ergodic set with parameters (kr,q,R) where $q = q_0(1 + 2\max_{g=1,\ldots,r} q_g)$.

Le Bras (1977) assumes that there is a primitive $r \times r$ state connection matrix, call it $\underset{\sim}{a}$, such that if $\underset{\sim}{x}$ is any multistate projection matrix, $\underset{\sim}{a} \leq c(\underset{\sim}{x})$. In Theorem 3.2, we require only that the set of all state connection matrices be an ergodic set. To see that this requirement is weaker, suppose the state connection matrix of a 3-regional population at any given time were either $\underset{\sim}{c}_1$ or $\underset{\sim}{c}_2$, where

$$\underset{\sim}{c}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} , \quad \underset{\sim}{c}_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and that the state connection matrix could change from time to time. The largest matrix that is elementwise less than $\underset{\sim}{c}_1$ and $\underset{\sim}{c}_2$ is $\underset{\sim}{I}$, the 3×3 identity matrix, which is not primitive. Thus the population just described is not covered by the results of Le Bras (1977). But it is readily checked that $\underset{\sim}{c}_1^2 > 0$, $\underset{\sim}{c}_2^2 > 0$, and $\underset{\sim}{c}_1\underset{\sim}{c}_2 > 0$, so $\{\underset{\sim}{c}_1, \underset{\sim}{c}_2\}$ is an ergodic set and the population described may be covered by Theorem 3.2.

Proof of Theorem 3.2. Let $Q = \max_{g=1,\ldots,r} q_g$. Then $H(1,Q)$ has strictly positive diagonal $k \times k$ submatrices. Then suppose some matrix $\underset{\sim}{x}^{(a)} \in X$ has $c_{gh}(\underset{\sim}{x}^{(a)}) = 1$. Then in the g,h submatrix of $H(1,Q)\underset{\sim}{x}^{(a)}$ there is at least one positive column. Hence $H(1,Q)\underset{\sim}{x}^{(a)}H'(1,Q)$, where $H'(1,Q)$ is any product of Q matrices from X, chosen independently of the Q factors in $H(1,Q)$, has a positive $k \times k$ submatrix in the g,h position. Thus wherever $c_{gh}(\underset{\sim}{x}^{(a)}) = 1$, there is a positive submatrix in the g,h position of $H(1,Q)\underset{\sim}{x}^{(a)}H'(1,Q) \equiv H_a(1,2Q+1)$, where the subscript a shows that $\underset{\sim}{x}^{(a)}$ is the $(Q+1)$th factor in this product of $2Q+1$ otherwise arbitrary matrices from X. Now let $\underset{\sim}{x}^{(a)},\ldots,\underset{\sim}{x}^{(q_0)}$ be any q_0 elements of X and form $H_a(1,2Q+1),\ldots,H_{q_0}(1,2Q+1)$ where the factors other than the $(Q+1)$th are arbitrary. By (iii), $c(\underset{\sim}{x}^{(a)})\ldots c(\underset{\sim}{x}^{(q_0)}) > 0$. Hence $H_a(1,2Q+1) \ldots H_{q_0}(1,2Q+1) > 0$. Since $\underset{\sim}{x}^{(\tilde{a})},\ldots,\underset{\sim}{x}^{(q_0)}$ are arbitrary, we have shown that an arbitrary product of $q_0(2Q+1)$ matrices from X is positive. This proves Theorem 3.2.

Condition (ii) of Theorem 3.2 permits an element of a diagonal submatrix of the multistate projection matrices to be 0 at some times and positive at others. For application to real matrices used for multiregional projection, it is desirable to weaken (ii) further.

If the k age classes include post-reproductive ages, then $\underset{\sim}{x}_{gg}$ need not be primitive. Suppose that the last age class with positive effective fertility is the same in every region; call this age class β. Assume positive survival proportions up to age β. Formally, suppose $\underset{\sim}{x}_{gh}(1,\beta) > 0$; $\underset{\sim}{x}_{gh}(1,j) = 0$, $j > \beta$; $\underset{\sim}{x}_{gh}(j+1,j) > 0$, $0 < j < \beta$. Then Ledent (1972) proved that a cogredient permutation of rows and columns can put $\underset{\sim}{x}$ in the form

$$\underset{\sim}{x}' = \begin{pmatrix} M & 0 \\ A & B \end{pmatrix} \tag{3.1}$$

where $\underset{\sim}{M}$ is $(r\beta) \times (r\beta)$ and $\underset{\sim}{B}$ is $r(k-\beta) \times r(k-\beta)$. The r^2 $\beta \times \beta$ submatrices $\underset{\sim}{M}_{gh}$ of $\underset{\sim}{M}$ are the northwest $\beta \times \beta$ submatrices of $\underset{\sim}{x}_{gh}$. The $(k-\beta) \times \beta$ submatrices $\underset{\sim}{A}_{gh}$ of $\underset{\sim}{A}$ are the southwest $(k-\beta) \times \beta$ submatrices of $\underset{\sim}{x}_{gh}$ and are zero everywhere except possibly for $\underset{\sim}{A}_{gh}(1,\beta) \geq 0$. The $(k-\beta) \times (k-\beta)$ submatrices $\underset{\sim}{B}_{gh}$ of $\underset{\sim}{B}$ are the southeast $(k-\beta) \times (k-\beta)$ submatrices of $\underset{\sim}{x}_{gh}$ and are zero except possibly for $\underset{\sim}{B}_{gh}(j+1,j) \geq 0$, $j = 1,\ldots,k-\beta-1$. The northeast $r\beta \times r(k-\beta)$ submatrix of $\underset{\sim}{x}'$ is zero. Ledent (1972) observed that if $\underset{\sim}{x}$ is a multiregional projection matrix for a real human population, then $\underset{\sim}{M}$ as described may be assumed to have primitive diagonal submatrices $\underset{\sim}{M}_{gg}$ and $c_{ij}(\underset{\sim}{M}) = 1$, $i,j = 1,2,\ldots,r$, so that $\underset{\sim}{M}$ is primitive by Feeney's (1971) result.

Theorem 3.3 generalizes Ledent's (1972) observation to inhomogeneous matrix products.

THEOREM 3.3. Let X be a set of multistate projection matrices for r states with k age classes. Each $\underset{\sim}{x} \in X$ can be partitioned into r^2 $k \times k$ submatrices $\underset{\sim}{x}_{gh}$. (i) Suppose there is an integer β, $1 \leq \beta < k$, such that, for $g,h = 1,\ldots,r$, each $k \times k$ submatrix $\underset{\sim}{x}_{gh}$ can be partitioned in the form

$$\underset{\sim}{x}_{gh} = \begin{pmatrix} \underset{\sim}{M}_{gh} & \underset{\sim}{0} \\ \underset{\sim}{A}_{gh} & \underset{\sim}{B}_{gh} \end{pmatrix} \tag{3.2}$$

where $\underset{\sim}{M}_{gh}$ is $\beta \times \beta$, $\underset{\sim}{B}_{gh}$ is $\gamma \times \gamma$ with $\beta+\gamma = k$ and the zero matrix $\underset{\sim}{0}$ is $\beta \times \gamma$. Moreover, suppose there exists $R > 0$ such that, for all $\underset{\sim}{x} \in X$ and all g, h = $1,\ldots,r$, $\min^+(\underset{\sim}{M}_{gh})/\max(\underset{\sim}{M}_{gh}) > R$. (ii) Suppose that, for each g, $\{\underset{\sim}{M}_{gg}; \underset{\sim}{x} \in X\}$ is a primitive set with parameters (β, q_g). (iii) Let $c_{g,h}^{(\underset{\sim}{M})}(\underset{\sim}{x}) = 1$ if $\underset{\sim}{M}_{gh}$ has at least one positive element and $c_{g,h}^{(\underset{\sim}{M})}(\underset{\sim}{x}) = 0$ if $\underset{\sim}{M}_{gh} = \underset{\sim}{0}_{\beta\times\beta}$, g, h $= 1,\ldots,r$. Then suppose that $\{c^{(\underset{\sim}{M})}(\underset{\sim}{x}); \underset{\sim}{x} \in X\}$ is a primitive set with parameters $(r, q_0, 1)$. (iv) For all g, h = $1,\ldots,r$, let $\underset{\sim}{A}_{gh}(1,\beta) > 0$; i.e. the northeast element of every $\underset{\sim}{A}_{gh}$ is positive. The other elements of $\underset{\sim}{A}_{gh}$ may be 0 or positive. (v) For all g, h = $1,\ldots,r$, let $\underset{\sim}{B}_{gh}$ be strictly lower triangular with positive subdiagonal, i.e.

$B_{gh}(i,j) = 0$, $i = 1,\ldots,\gamma$; $j = i,\ldots,\gamma$; $B_{gh}(i+1,i) > 0$, $i = 1,\ldots,\gamma-1$.
Then X is an exponentially contracting set.

We cannot conclude X is an ergodic set because the northeast $\beta \times \gamma$ corner of every $k \times k$ submatrix will always be 0. We have not assumed any quantitative restrictions on the elements of A_{gh} and B_{gh}. When each submatrix x_{gh} is interpreted as a Leslie matrix, (iv) assumes a positive proportion surviving from age class β to $\beta+1$ and (v) assumes positive proportions surviving from age class $\beta+1$ to age class k. These conditions are met by human populations.

Proof of Theorem 3.3. By a cogredient permutation of rows and columns, each $x \in X$ takes the form x' in (3.1) described by Ledent (1972). Then assumptions (ii) and (iii) impose on the set of all matrices that occupy the position of M in (3.1) exactly the same conditions that assumptions (ii) and (iii) of Theorem 3.2 impose on all $x \in X$. Consequently, by Theorem 3.2 $\{M; x \in X\}$ is an ergodic set. If $H'(1,q)$ is the cogrediently permuted form of the product of q arbitrary $x \in X$, then the northwest $r\beta \times r\beta$ submatrix of $H'(1,q)$ is positive for $q \geq Q = \max[\gamma, q_0(1+2\max_{g=1,\ldots,r} q_g)]$, again by Theorem 3.2. Also since the product of any γ strictly lower triangular $\gamma \times \gamma$ matrices is 0, columns $r\beta+1, r\beta+2,\ldots,rk$ are 0 in $H'(1,q)$ for $q \geq Q$. It remains only to describe what happens to the southwest $r\gamma \times r\beta$ submatrix of $H'(1,q)$, $q \geq Q$, in the position corresponding to A in (3.1). Assumptions (iv) and (v) imply that, as q increases from $Q+1$ to $Q+\gamma$, the minimum number of positive elements in the βth column of each $\gamma \times \beta$ submatrix of the $r\gamma \times r\beta$ southwest corner of $H'(1,q)$ increases from 1 to γ. Thus as q increases from $Q+1$ to $Q+\gamma$, each of columns $h\beta$, $h = 1,\ldots,r$ has at least $r, 2r, \ldots, \gamma r$ positive elements. Thus for $q \geq Q+\gamma$, $H'(1,q)$ has at least r positive columns in positions $h\beta$, $h = 1,\ldots,r$; has 0 everywhere in columns $r\beta+1,\ldots,rk$; and is strictly positive in the intersection of rows and columns $1,\ldots,r\beta$.

The proof of Theorem 6 of Cohen (1979, p. 362) therefore applies to $H(1,q)$, $q \geq Q+\gamma$ and shows that X is an exponentially contracting set. This proves Theorem 3.3.

When the states of a multistate projection matrix correspond to geographical regions, to being employed or unemployed, or to being married or unmarried, it is reasonable to suppose that, in the course of time, there is a positive migration from each state to every other state, in age classes prior to the last age of reproduction, as in Theorem 3.3 (iii). But when the states are {without high school diploma; with high school diploma} or {never married; ever married}, some states cannot be re-entered, once they are left. Even so, there are conditions on multistate projection matrices sufficient to guarantee that a set of them will be exponentially contracting. For simplicity, we describe here only the special case of $r = 2$ states.

THEOREM 3.4. Let X be a set of $2k \times 2k$ two-state projection matrices with k age classes. Partition each $\underset{\sim}{x} \in X$ into four $k \times k$ submatrices

$$x = \begin{pmatrix} \underset{\sim}{x}_{11} & \underset{\sim}{x}_{12} \\ \underset{\sim}{x}_{21} & \underset{\sim}{x}_{22} \end{pmatrix} .$$

(i) Let $\{\underset{\sim}{x}_{11}; \underset{\sim}{x} \in X\}$ be an ergodic set with parameters (k, q_1, R_1). (ii) Let $\underset{\sim}{x}_{12} = \underset{\sim}{0}_{k\times k}$, for all $\underset{\sim}{x} \in X$; there are no transitions from state 2 to state 1. (iii) Assume there is at least one positive element in each row of $\underset{\sim}{x}_{21}$ for all $\underset{\sim}{x} \in X$. (This means there are positive flows from state 1 to every age class of state 2.) (iv) Suppose there exist constants K_1, K_2, K_3 such that $0 \leq kK_1 \leq 1$, $0 < K_2 \leq K_3 < \infty$, and for all $\underset{\sim}{x} \in X$, (a) $0 \leq \max(\underset{\sim}{x}_{22})/\min^{+}(\underset{\sim}{x}_{11}) \leq K_1$; (b) $K_2 \leq \min^{+}(\underset{\sim}{x}_{21})/\max(\underset{\sim}{x}_{11})$; (c) $\max(\underset{\sim}{x}_{21})/\min^{+}(\underset{\sim}{x}_{11}) \leq K_3$. Then X is an exponentially contracting set.

Proof of Theorem 3.4. Theorem 3.4 is just a restatement, in the context of multistate projection matrices, of Theorem 5 of Cohen (1979, p. 359).

In applying Theorem 3.4 to real sets of multistate projection matrices, condition (i) can usually be assured by truncating after the largest age class with positive fertility. Condition (iii) assumes positive transitions from state 1 to every age class of state 2. For states defined in terms of education, employment, or marriage, very young children usually do not change states.

If, for example, a 5-year age class and time unit are used, some newborn individuals will change educational, employment or marriage status after 20 years, so all products of 4 matrices from X will have at least one positive element in each row of $\underset{\sim}{x}_{21}$ corresponding to young ages. Thus X can be replaced by all products of 4 matrices from X. If adults past a certain age do not change states, these age classes can be truncated, as is commonly done for post-reproductive age classes. Condition (iv,a) requires that the largest survival and effective fertility coefficients in state 2 all be small compared to the smallest coefficients in state 1. Thus the dynamics of state 1 dominate the projection under the conditions assumed in Theorem 3.4.

Le Bras (1977, p. 274) mentions qualitatively the case we consider in Theorem 3.4, but he offers no analysis of it.

None of Theorems 3.1 to 3.4 requires the set X of multistate projection matrices to be finite, or even countably infinite.

4. STOCHASTIC ERGODIC THEOREMS

So far, we have assumed that the sequence $\underset{\sim}{x}(t)$ of multistate projection matrices was chosen by some deterministic mechanism. Now we assume that the sequence $\underset{\sim}{x}(t)$ represents the sample path of a Markov chain. We have chosen a Markov chain as the process governing $\underset{\sim}{x}(t)$ because a Markov chain can represent sequential dependence of $\underset{\sim}{x}(t+1)$ on $\underset{\sim}{x}(t)$, yet is simple enough to be analyzed in detail. Whether the dependence of $\underset{\sim}{x}(t)$ on the past is really Markovian remains to be determined.

We shall proceed naively, without specifying which sets and functions are assumed to be measurable. Readers who recognize the need for such qualifications can supply them from the results already obtained for a single-regional age-structured population (Cohen, 1977a,b).

We recall some definitions from the theory of finite Markov chains. Following Kemeny and Snell (1960), a Markov chain is ergodic if it is possible to go, directly or indirectly, from any state to any other state. A cyclic or periodic chain is an ergodic chain in which each state can only be entered at certain periodic intervals. A regular chain is an ergodic chain that is not cyclic.

THEOREM 4.1. Let X be an exponentially contracting set containing s (s finite) multistate projection matrices $\tilde{x}^{(1)},\ldots,\tilde{x}^{(s)}$, each of which is $(kr)\times(kr)$. Let $\tilde{P}[\tilde{x}(t+1) = \tilde{x}^{(j)} \mid \tilde{x}(t) = \tilde{x}^{(i)}] = p_{ij}$, $i,j = 1,\ldots,s$, where $\tilde{P} = (p_{ij})$ is the (primitive) transition probability matrix of a regular Markov chain. Let $Y = \{y;\ y$ is a kr-vector, $y \geq 0$ and $\|y\| = 1\}$, where $\|y\| = \sum_i |y_i|$. For any kr-vector $Y(0) > 0$, define $Y(t)$ by (2.1) and define $y(t) = Y(t)/\|Y(t)\| \in Y$. Then:

(i) The bivariate process $(\tilde{x}(t), y(t))$ is a Markov chain (with uncountably many states) on the state space $X \times Y$. If T is the transition probability function of the bivariate chain $(\tilde{x}(t), y(t))$, that is, $T(\tilde{x}^{(i)}, y, \tilde{x}^{(j)}, B)$ is the probability of a transition from $(\tilde{x}^{(i)}, y)$ into $(\tilde{x}^{(j)}, B)$, then T may be expressed explicitly in terms of $\tilde{P}$ and of matrix multiplication as $T(\tilde{x}^{(i)}, y, \tilde{x}^{(j)}, B) = p_{ij} I_B(\tilde{x}^{(j)} y / \|\tilde{x}^{(j)} y\|)$, and for $B \subset Y$, $I_B(y) = 1$ if $y \in B$, $I_B(y) = 0$ if $y \notin B$.

(ii) There is a limiting probability distribution $F(A,B)$ defined on subsets A of X and subsets B of Y such that $\lim_{t\to\infty} \tilde{P}[\tilde{x}(t) \in A,\ y(t) \in B] = F(A,B)$, independent of initial conditions. F may be calculated numerically by solving the renewal equation

$$F(\tilde{x}^{(j)}, B) = \sum_{i=1}^{s} \int_{y\in Y} F(\tilde{x}^{(i)}, dy)\, T(\tilde{x}^{(i)}, y, \tilde{x}^{(j)}, B) \quad .$$

(iii) Let $Y_\delta = \{y;\ y$ is a kr-vector, $y \geq 0$, and $\min_i y_i / \max_j y_j \geq \delta\}$. If X is an ergodic set, let $\delta = 0$. If X is not an ergodic set (but still is exponentially contracting, as assumed at the outset), fix $0 < \delta < 1$. Then there exist positive constants a (depending on δ and X) and b (depending only on X) such that, for any initial census $Y(0)$ in Y_δ, any initial projection matrix $\tilde{x}^{(i)}$ in X, and any subset B of Y,

$$\left| \tilde{P}[\tilde{x}(t) = \tilde{x}^{(j)}, y(t) \in B \mid \tilde{x}(1) = \tilde{x}^{(i)}, y(0) = Y(0)/\|Y(0)\|] - F(\tilde{x}^{(j)}, B) \right| < a e^{-bt}.$$

(iv) For a scalar or vector-valued function g with domain $X \times Y$,

$$\lim_{t\to\infty} t^{-1} \sum_{\theta=1}^{t} g(\underset{\sim}{x}(\theta), y(\theta)) = \sum_{i=1}^{s} \int_Y g(\underset{\sim}{x}^{(i)}, y) F(\underset{\sim}{x}^{(i)}, dy)$$

whenever the right side of the equation exists.

(v) There is a constant $\lambda > 0$ such that, for any initial census $Y(0)$ and for almost all sample paths of the $\underset{\sim}{x}(t)$ chain,

$$\log \lambda = \lim_{t\to\infty} t^{-1} \log \| Y(t) \| = \lim_{t\to\infty} t^{-1} E[\log \| Y(t) \|] \quad .$$

This $\log \lambda$ is the asymptotic almost-sure growth rate of Furstenberg and Kesten (1960). A formula for calculating $\log \lambda$ is

$$\log \lambda = \sum_{i=1}^{s} \sum_{j=1}^{s} \int_Y \log(\| \underset{\sim}{x}^{(j)} y \|) p_{ij} F(\underset{\sim}{x}^{(i)}, dy) \quad .$$

If $c_{(i)}$ is the smallest of the column sums of $\underset{\sim}{x}^{(i)}$ and $c^{(i)}$ is the largest of the column sums of $\underset{\sim}{x}^{(i)}$, then

$$-\infty < \sum_{i=1}^{s} \pi_i \log c_{(i)} \leq \log \lambda \leq \sum_{i=1}^{s} \pi_i \log c^{(i)} < \infty \quad ,$$

where

$$\pi_i = \lim_{t\to\infty} \underset{\sim}{P}[\underset{\sim}{x}(t) = \underset{\sim}{x}^{(i)}] \quad .$$

(vi) There is a constant $\mu \geq \lambda > 0$ such that

$$\log \mu = \lim_{t\to\infty} t^{-1} \log E\| Y(t) \| \quad ,$$

where

$$E\| Y(t) \|$$

is just the average (over all sample paths) total population size of the census at time t. Thus μ is the asymptotic growth rate of the average population size, while λ is the average of the growth rates along each sample path. μ is the dominant eigenvalue of the $(skr) \times (skr)$ matrix $\underset{\sim}{P}^T \otimes X = \underset{\sim}{M}$, defined as consisting of s^2 submatrices $\underset{\sim}{M}_{ij} = p_{ji} \underset{\sim}{x}^{(i)}$, each of order $kr \times kr$.

When X contains only a single matrix $\underset{\sim}{x}$, λ and μ are both the dominant eigenvalue of $\underset{\sim}{x}$.

(vii) If X is an ergodic set, then all regions grow asymptotically at the same rate, i.e., for $i,j = 1,2,\ldots,kr$,

$$\log \mu = \lim_{t\to\infty} t^{-1} \log E(\underset{\sim}{x}_{ij}(t)), \log \lambda = \lim_{t\to\infty} t^{-1} E(\log \underset{\sim}{x}_{ij}(t)) \quad .$$

(viii) The asymptotic variance in the logarithm of the increase per unit time in population size is

$$\sigma^2 = \lim_{t\to\infty}(E[\log(\|Y(t+1)\|/\|Y(t)\|)]^2 - \{E[\log(\|Y(t+1)\|/\|Y(t)\|)]\}^2)$$

where

$$\lim_{t\to\infty} E[\log(\|Y(t+1)\|/\|Y(t)\|)]^2 = \sum_{i=1}^{s} \sum_{j=1}^{s} \int_Y [\log\|\underset{\sim}{x}^{(j)} y\|]^2 p_{ij} F(\underset{\sim}{x}^{(i)}, dy)$$

and

$$\lim_{t\to\infty} E[\log(\|Y(t+1)\|/\|Y(t)\|)] = \log \lambda \quad .$$

(Note that σ^2 is <u>not</u> the variance of $\lim_{t\to\infty} t^{-1}\log\|Y(t)\|$. For all sample paths, except those belonging to a set of probability 0, $\lim_{t\to\infty} t^{-1}\log\|Y(t)\|$ is the constant $\log \lambda$ and the variance of $\lim_{t\to\infty} t^{-1}\log\|Y(t)\|$ is 0.)

(ix) Let v be a real kr-vector. (If every element of v is 1, $v^T Y(t)$ is the total population size at time t. If v^T contains the labor-force participation rates, assumed constant, by age and state, $v^T Y(t)$ is the labor force at time t.) Lange and Hargrove (1980) give explicit recursive formulas for computing the right side of

$$\mathrm{Var}(v^T Y(t)) = (v^T \otimes v^T)[E(Y(t) \otimes Y(t)) - E(Y(t)) \otimes E(Y(t))]$$

where $\otimes$ is the ordinary tensor or Kronecker product. Thus the mean and variance of any homogeneous linear function of the census Y(t) can be calculated at any time t.

The proof of this theorem is so close to the proofs in Cohen (1977a,b) and Lange and Hargrove (1980) that we do not repeat the details. The only significant change is that we have replaced the requirement that X be an ergodic set by the conditions that X be an exponentially contracting set and, in part (iii), $Y(0) \in Y_\delta$.

5. EXTENSIONS AND APPLICATIONS

In this section I review briefly some possible extensions of this multiregional stochastic theorem which have already been worked out in the single-regional case, and then indicate what it would take to put the theorem to work with real data.

The Markov chain that governs the succession of multiregional projection matrices could be extended from a finite-state chain to a countably infinite (Cohen, 1976) or uncountably infinite (Cohen, 1977a) chain. The restriction to a homogeneous chain could be dropped (Cohen, 1977a,b) at the price of losing an invariant long run distribution of vital rates and census structure. If the chain is homogeneous ergodic but periodic, instead of regular, the distributions of vital rates and census structure converge in Cesaro sums (Lange, 1979). The assumption of a single sex could be replaced by a female-dominant two-sex model and immigration into the multiregional population could be considered (Lange and Hargrove, 1980). The interaction of demographic with exogenous environmental variables could be considered (Land, 1980). Although the explicit formulas for calculating long-run growth rates depend on a Markovian assumption, the existence of the long-run growth rates can be proved if a stationary stochastic process, not necessarily Markovian, is assumed to choose successive projection matrices (Lange and Holmes, 1980).

I have already described (1976, pp. 335-336; 1977a, pp. 24-25) how this stochastic model suggests a scheme for the analysis of historical data and for the construction of probabilistically interpretable projections. Here I mention just the data requirements. To estimate the Markov chain, one requires a sequence of observed projection matrices. Ten would be a minimum. The aim would be first to reduce the dimensionality of such arrays of numbers by fitting parametric models to the fertility, mortality, and migration rates, then to fit some Markov process to the model parameters. To prepare projections, one requires in addition a current census by age and region.

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