WORKING PAPER

MAINTAINING THE POSITIVE DEFINITENESS OF THE MATRICES IN REDUCED SECANT METHODS FOR EQUALITY CONSTRAINED OPTIMIZATION

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FOREWORD

This paper proposes an algorithm for minimizing a function f on $\mathbb{R}^n$ in the presence of m equality constraints c that locally is a reduced secant method. The local method is globalized using a nondifferentiable augmented Lagrangian whose decrease is obtained by both a longitudinal search that decreases mainly f and a transversal search that decreases mainly $||c||$.

The main objective of the paper is to show that the longitudinal path can be designed in order to maintain the positive definiteness of the reduced matrices by means of the positivity of $\gamma_k^T \delta_k$, where γ_k is the change in the reduced gradient and δ_k is the reduced longitudinal displacement.

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MAINTAINING THE POSITIVE DEFINITENESS OF THE MATRICES IN REDUCED SECANT METHODS FOR EQUALITY CONSTRAINED OPTIMIZATION *

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1. Introduction

We consider here the problem of minimizing a real-valued function f defined on an open convex set ω in $\mathbb{R}^n$ subject to m nonlinear equality constraints c $(m < n)$:

$$\min \{ f(x) : x \in \omega \, , \, c(x) = 0 \} \, . \tag{1.1}$$

We shall suppose that the $m \times n$ Jacobian matrix of the constraints at a point x in ω, namely $A(x)$, is surjective, i.e. has full rank m. Then, if x_* is a local solution of (1.1), there exists a unique Lagrange multiplier λ_* in $\mathbb{R}^m$, such that the following first order optimality conditions are satisfied (see Fletcher (1981), for example):

$$\begin{cases} c(x_*) = 0 \, , \\ \nabla f(x_*) + A(x_*)^T \lambda_* = 0 \, , \end{cases} \tag{1.2}$$

where $\nabla f(x_*)$ denotes the gradient vector of f at x_*, an $n \times 1$ matrix, and $A(x_*)^T$ is the transposed matrix of $A(x_*)$.

Locally, the faster methods for solving (1.1) amount to finding solutions of (1.2), which correspond to stationary solutions of the original problem. Two classes of local methods may be distinguished. The first class is formed of those algorithms whose step is an approximation of the Newton step for solving (1.2). Among them are the quasi-Newton methods, which may be introduced as follows. The Jacobian matrix of (1.2) at (x_*,λ_*) writes

$$J_* := \begin{pmatrix} A(x_*) & O \\ L_* & A(x_*)^T \end{pmatrix} ,$$

where L_* is the Hessian according to x of the Lagrangian $l(x,\lambda) := f(x) + \lambda^T c(x)$ evaluated at (x_*,λ_*). If J_* is approximated by

$$J_k := \begin{pmatrix} A(x_k) & O \\ L_k & A(x_k)^T \end{pmatrix} ,$$

where L_k is a symmetric matrix of order n and if we denote by $\nabla_x l(x_k,\lambda_k)$ the gradient according to x of the Lagrangian, quasi-Newton methods write

$$\begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ \lambda_k \end{pmatrix} - J_k^{-1} \begin{pmatrix} c(x_k) \\ \nabla_x l(x_k,\lambda_k) \end{pmatrix} .$$

Note that if L_k is positive definite, or only positive definite in $Ker\ A(x_k)$, the kernel of $A(x_k)$, i.e. $\xi^T\ L_k\ \xi > 0$ for all non zero ξ in $Ker\ A(x_k)$, then J_k is non singular and the previous iteration is well defined. This method is called the *Successive Quadratic Programming* (SQP) method because $x_{k+1} = x_k + d_k^{SQP}$ where d_k^{SQP} is obtained by solving successively in d the following quadratic programs:

$$\left\{\begin{array}{l} \min\ \nabla f(x_k)^T\ d + \dfrac{1}{2}\ d^T\ L_k\ d\ , \\ s.t.\ \ d \in I\!R^n\ ,\ c(x_k) + A(x_k)\ d = 0 \end{array}\right. \tag{1.3}$$

and $\lambda_{k+1} = \lambda_{k+1}^{SQP}$, the associated multiplier. In this algorithm, L_k is updated at each iteration. This method has been extensively studied since the papers by Wilson (1963), Han (1976) and Powell (1978) and we refer to Powell (1986) for a state of the art on the subject.

The second class of methods is based on the fact that the dimension of problem (1.1) is not n but is equal to the dimension $n-m$ of the manifold $M(x_*) := c^{-1}(0)$ on which f is minimized. Therefore, one may expect to find secant methods in which the updated matrices are of order $n-m$. This is certainly a realistic expectation if we impose the iterates x_k to belong to the manifold $M(x_*)$, i.e. $c(x_k) = 0$ for all k. Indeed, in this case, c being a submersion, $M(x_*)$ is a submanifold of $I\!R^n$ (see e.g. Leborgne (1982)) and there exists a smooth parametric representation ξ of $M(x_*)$ in a neighborhood V of x_*, i.e. a function $\xi : U \longrightarrow M(x_*) \bigcap V$ such that $c(\xi(u)) = 0$ for all u in the open set U of $I\!R^{n-m}$. Therefore, working on the set U to minimize $f(\xi(u))$ will give the expected algorithm. But it is usually unrealistic to impose $c(x_k) = 0$ and fortunately, this is not necessary. As far as we know, the first reduced secant methods (reduced because, for example, the order of the updated matrices in $n-m$ rather than n in the SQP method) without the feasibility condition ($c(x_k) = 0$) are due to Gabay (1982,b) and Coleman and Conn (1982,a,b). Theoretically, the method proposed by Coleman and Conn seems better than the method studied by Gabay. The convergence of the latter is, indeed, only superlinear in two steps in general (see Coleman and Conn (1982,a and 1984), Byrd (1985) and Yuan (1985)), that is to say:

$$||x_{k+1}-x_*||\ /\ ||x_{k-1}-x_*|| \longrightarrow 0\ , \tag{1.4}$$

while the convergence of the former has been proved to be superlinear (in one step) (see Byrd (1984) and Gilbert (1986,a,c)), that is to say:

$$||x_{k+1}-x_*||\ /\ ||x_k-x_*|| \longrightarrow 0\ . \tag{1.5}$$

This is a better rate of convergence than the rate (1.4). Note that this rate of convergence can also be obtained by using the SQP method but with the necessity of updating a matrix of order n. Therefore, reduced secant methods become competitive and sometimes the only one usable when the number m of constraints is large while the number $n-m$ of parameters remains reasonable.

In this work, we shall focus on some aspects of the method proposed by Coleman and Conn. But first, what is this method? One way to introduce the algorithm is to say that it tries to solve the system of optimality (1.2) by considering both equations separately and successively. So, starting from an estimate (x_k,λ_k) of (x_*,λ_*), the next iterate (x_{k+1},λ_{k+1}) is calculated in two steps (see Gilbert (1987)):

$$y_k := x_k - R_k\, c(x_k) =: x_k + r_k\ , \tag{1.6}$$

$$x_{k+1} := y_k - Z(y_k)^- \, H_k\, g(y_k) =: y_k + t_k\ , \tag{1.7}$$

$$\lambda_{k+1} := -\, A(y_k)^{-T}\, \nabla f(y_k) + A(y_k)^{-T}\, L_k\, Z(y_k)^-\, H_k\, g(y_k)\ . \tag{1.8}$$

In (1.6), R_k is a restoration operator, an $n \times m$ *injective* matrix asymptotically close to $A(x_*)^-$, a right inverse of $A(x_*)$. Here, we shall take $R_k := A(x_k)^-$ a right inverse of $A(x_k)$, although $R_k := A(y_{k-1})^-$, which avoids the linearization of the constraints at x_k, is also possible without destroying the superlinear convergence but is more tricky to handle. In (1.7), $Z(y_k)^-$ is an $n \times (n-m)$ matrix whose columns form a basis of $Ker\ A(y_k)$, which is the tangent space to $M(y_k) := c^{-1}(c(y_k))$ at y_k. Later, we shall say that $Z(y)^-$ is *a basis of* $Ker\ A(y)$, shortly. Hence

$$A(y)\, Z(y)^- = 0\ , \tag{1.9}$$

for all y in ω and $Z(y)^-$ is *injective*. In (1.7) again, H_k is a symmetric matrix of order $n-m$ that will be updated so as to remain positive definite (the main concern of this paper) and so as to have

$$(G_k - G_*)\, Z(y_k)\, t_k = o(\|t_k\|)\ , \tag{1.10}$$

where $Z(y)$ is the unique $(n-m) \times n$ matrix satisfying

$$Z(y)\, Z(y)^- = I \quad \text{in} \quad \mathbb{R}^{(n-m)\times(n-m)}\ , \tag{1.11}$$

$$Z(y)\, A(y)^- = 0 \quad \text{in} \quad \mathbb{R}^{(n-m)\times m}\ , \tag{1.12}$$

$G_k := H_k^{-1}$ and G_* is the *reduced Hessian of the Lagrangian* defined by

$$G_* := Z(x_*)^{-T}\, L_*\, Z(x_*)^-\ . \tag{1.13}$$

In (1.7), at last, $g(y_k)$ is the *reduced gradient* of f at y_k and is defined by

$$g(y) := Z(y)^{-T}\, \nabla f(y)\ .$$

Finally, in (1.8), L_k is an approximation of L_*. For more details on this formalism, and for examples of choices for $A(y)^-$ and $Z(y)^-$, we refer to Gabay (1982,a).

The algorithm (1.6)-(1.8) calls for some comments. First, note that λ_k does not intervene in the calculation of x_{k+1} and λ_{k+1}. Therefore, from the superlinear convergence of the sequence (x_k,λ_k) (together), we can deduce the superlinear convergence of (x_k), while for (λ_k) we get

$$||\lambda_{k+1}-\lambda_*|| \;/\; ||x_k-x_*|| \longrightarrow 0 \,. \tag{1.14}$$

We also see that the sequence (x_k) can be generated by (1.6)-(1.7) independently of the sequence (λ_k). We shall see, indeed, that the update of H_k does not require the knowledge of (λ_k). In any case, the sequence (x_k) can be obtained (see Gilbert (1986,c)) by solving the system

$$\begin{cases} c(x_*) = 0 \,, \\ g(x_*) = 0 \,, \end{cases}$$

whose second equation is obtained by projecting the second equation of (1.2) on the tangent space $Ker\ A(x_*)$, i.e. by multiplying it to the left by $Z(x_*)^{-T}$. Therefore, the algorithm (1.6)-(1.7) is a reduced method for (x_k) because the only matrix to update is H_k which is of order $n-m$. But, this is not true any more if we want to calculate (λ_k), because L_k is present in (1.8). We have seen that, locally, this calculation is not necessary. However, in a global framework, some estimate of λ_* is useful. Therefore, we shall avoid the need of generating L_k by taking the following estimate:

$$\lambda(y) := -\ A(y)^{-T}\,\nabla f(y) \,, \tag{1.15}$$

whose value at $y = x_*$ is λ_*.

The local algorithm (1.6)-(1.7) is usually globalized by using a penalty function θ of the form:

$$\theta(x) := f(x) + \varphi(c(x)) \,, \tag{1.16}$$

where $\varphi : I\!R^m \longrightarrow I\!R_+$ is generally convex. If this is the case, one may calculate $\theta'(x\ ;\ t)$, the directional derivative of θ at x is a direction t and, in particular, one finds

$$\theta'(y_k\ ;\ t_k) = \nabla f(y_k)^T\, t_k + \varphi'(c(y_k)\ ;\ A(y_k)\, t_k) = -\ g(y_k)^T H_k\, g(y_k) \,, \tag{1.17}$$

because $t_k \in Ker\ A(y_k)$. This shows that it is interesting to maintain H_k positive definite in order that t_k will be a decent direction of θ at y_k.

Our main objective in this paper is to design a globally and superlinearly convergent algorithm that locally reduces to the method (1.6)-(1.7) and that maintains the matrices H_k positive definite, updating them by the *inverse BFGS formula* (see e.g. Dennis and Moré (1977)):

$$H_{k+1} = \left[I - \frac{\delta_k\, \gamma_k^T}{\gamma_k^T\, \delta_k} \right] H_k \left[I - \frac{\gamma_k\, \delta_k^T}{\gamma_k^T\, \delta_k} \right] + \frac{\delta_k\, \delta_k^T}{\gamma_k^T\, \delta_k} \,, \tag{1.18}$$

which we shall refer to by $H_{k+1} = \overline{BFGS}(H_k, \gamma_k, \delta_k)$. In this formula, γ_k will be the change in the reduced gradient (a vector in $\mathbb{R}^{n-m}$) when making a certain displacement and δ_k is the corresponding "reduced" displacement (also a vector in $\mathbb{R}^{n-m}$). Then, it is well known that H_k transmits its positive definiteness to H_{k+1} if and only if

$$\gamma_k^T \delta_k > 0 . \tag{1.19}$$

We shall aim to satisfy this condition at each iteration in our algorithm.

Before outlining hereafter the solutions developed further, which are valid in the framework of reduced methods, let us recall briefly how the positive definiteness of the updated matrices is maintained both in the SQP method and in reduced methods.

A similar situation occurs when the SQP method is globalized using a penalty function of the form (1.16). According to Han (1977), the displacement d_k^{SQP} is a descent direction of the l_1 penalty function ($\varphi(\bullet) = ||\bullet||_1$, the l_1 norm, in (1.16)) if L_k in (1.3) is positive definite together with other conditions. Therefore, when L_k is updated by the *direct BFGS formula*:

$$L_{k+1} = L_k - \frac{L_k \delta_k \delta_k^T L_k}{\delta_k^T L_k \delta_k} + \frac{\gamma_k \gamma_k^T}{\gamma_k^T \delta_k} , \tag{1.20}$$

L_{k+1} will be positive definite if L_k is positive definite and if $\gamma_k^T \delta_k$ is positive. Because L_k has to approximate L_*, γ_k is the change in the gradient of the Lagrangian and we take usually

$$\gamma_k := \nabla_x l(x_{k+1}^{SQP}, \lambda_{k+1}^{SQP}) - \nabla_x l(x_k, \lambda_{k+1}^{SQP}) ,$$
$$\delta_k := x_{k+1}^{SQP} - x_k = \rho_k d_k^{SQP} ,$$

where ρ_k is some step-size given by a line search on the l_1 penalty function. Unfortunately, it may happen that the positivity of $\gamma_k^T \delta_k$ cannot be realized for some x_{k+1}^{SQP}, that is to say for some value of the step-size (d_k^{SQP} is supposed to be fixed and given by the quadratic subprogram), because the Lagrangian is not necessary bounded from below and may have a negative curvature in the direction d_k^{SQP}, even locally. This has led Powell (1978,a,b,c) to propose to change γ_k in (1.20) by some convex combination $\bar{\gamma}_k$ of γ_k and $L_k \delta_k$ in order to have $\bar{\gamma}_k^T \delta_k$ positive. However, this strategy does not seem to give always good results, in particular, L_k may become ill-conditioned (see Powell (1984)). This leaves the field open to other suggestions.

The papers analyzing the update of matrices in reduced methods are due to Coleman and Conn (1984), Nocedal and Overton (1985) and Gilbert (1986,a). In the first paper the analysis is local and, as we shall see, (1.19) is automatically satisfied close to optimal points satisfying the usual second order optimality conditions. The analysis of Nocedal and Overton is devoted to the algorithm studied by

Gabay (1982,b) and is also local. At last, the analysis in Gilbert (1986,a) is global but the reduced matrix is not updated if condition (1.19) is not satisfied. This does not prevent superlinear convergence from occurring because asymptotically (1.19) is satisfied. However, even far from the solution it may be interesting to update the matrix in order to improve the convergence. So one possibility is to use Powell's modification of γ_k, another one is proposed in this paper.

The paper is organized as follows. In *Section 2,* we specify the notations and state the hypotheses. In *Section 3,* we discuss the solutions adopted to realize condition (1.19) along the *longitudinal displacement* governed by the tangent step t_k. On the one hand, it is detailed how a step-size selection procedure attributed to Wolfe (1969) can be used to obtain (1.19) when the displacement is done on the manifold $M(y_k)$. On the other hand, a counter-example will show that a simple search along the direction t_k cannot assure Wolfe's criteria to be satisfied in general. However, a median solution can be obtained by using the algorithm of Lemaréchal (1981) for Wolfe's criteria in unconstrained optimization. Here, a change in the direction of search is made each time an unfruitful attempt to realize (1.19) is done. Therefore, the longitudinal path of search becomes piecewise linear, approximating roughly an "ideal" path on the manifold $M(y_k)$. The analysis in Section 3 is done using a penalty function having the general form (1.16). In *Section 4,* however, we insert the longitudinal search of Section 3 in a globally convergent algorithm by using tools that are now well developed in the specific literature. In particular, we motivate our choice of a nondifferentiable augmented Lagrangian

$$l_p(x,\mu) := f(x) + \mu^T c(x) + p \, ||c(x)|| \, , \tag{1.21}$$

where $||\bullet||$ is a norm on $\mathbb{R}^m$, as a merit function by the necessity to have a unit longitudinal step-size asymptotically, being inspired in that direction by the work of Bonnans (1984). We shall also be more specific about the *transversal displacement* which consists of a simple linear search from x_k in the direction r_k, using Armijo's technique on the same penalty function (1.21). Finally, we give a theorem showing the global convergence of the algorithm.

2. Hypotheses and notations

We shall suppose that ω is a convex open set of $\mathbb{R}^n$. The convexity of ω is not essential but it is assumed to discard technical problems when Taylor's theorem is used. On the other hand, assuming ω open is essential because we do not consider here a problem with general constraints or inequality constraints. ω will be the set where nice properties of f and c are encountered. Usually, it will not be possible to take $\omega = \mathbb{R}^n$.

We shall suppose that on ω, f and c are sufficiently smooth, three times continuously differentiable will be enough, and that their derivatives are bounded, which can be satisfied if ω is bounded and small enough. Later, we shall suppose that the sequences (x_k) and (y_k) remain in ω, so, this supposes implicitly, more or less, the boundedness of these sequences. We shall also suppose that c is a *submersion* on ω, i.e. that $A(x)$ is surjective for all x in ω. This is a rather strong hypothesis but a useful one because it allows to make a decomposition of the space $\mathbb{R}^n$ at each point x of ω in $Ker\ A(x) = R(Z(x)^-)$ (see (1.9)) and $Ker\ Z(x) = R(A(x)^-)$ (see (1.12)), which are complementary subspaces. Using (1.9), (1.11) and (1.12), we get

$$I = A(x)^- \, A(x) + Z(x)^- \, Z(x) \ \text{ in } \ \mathbb{R}^{n\times n} \, . \tag{2.1}$$

We shall also suppose that this decomposition is made in a smooth way. More precisely, the function

$$x \longrightarrow (A(x)^-, Z(x)^-)$$

will be supposed twice continuously differentiable on ω and, as well as its derivatives, bounded on ω. Because $Z(x) = [0\ I_{n-m}]\,[A(x)^-\ Z(x)^-]^{-1}$, the function $x \longrightarrow Z(x)$ will also have the same property. This may also appear as a strong hypothesis if ω is large, but it can be satisfied in a neighborhood of a solution x_* if some qualification hypothesis (A_* surjective) is satisfied. On this question, we refer to Byrd and Schnabel (1986).

We shall denote by x_* a *solution of problem (1.1)*, i.e. a local minimizer satisfying the standard second order sufficient conditions of optimality (see Fletcher (1981), for instance). Therefore we shall suppose the existence of a Lagrange multiplier λ_* in $\mathbb{R}^m$ such that (1.2) is satisfied and such that the Hessian of the Lagrangian at (x_*,λ_*) is positive definite in the tangent space $Ker\ A(x_*)$. In other words, G_* given in (1.13) will be supposed positive definite.

We shall denote by $||\bullet||$ any norm on $\mathbb{R}^n$ or $\mathbb{R}^m$ (and not necessary the same norms on both spaces) and by $||\bullet||_D$ the dual norm for the Euclidian scalar product, i.e. $||v||_D := \sup \{ \, v^T u : ||u|| \le 1 \, \}$. Matrix norms will be supposed subordinated to the vector norms, i.e. $||A|| := \sup \{ \, ||Au|| : ||u|| \le 1 \, \}$. If (u_k) is a sequence of vectors and (α_k) and (β_k) are two sequences of positive numbers, we shall note $u_k = O(\alpha_k)$ when $(||u_k||/\alpha_k)$ is bounded, $u_k = o(\alpha_k)$ when $(||u_k||/\alpha_k)$ converges to zero and $\alpha_k \sim \beta_k$ when $\alpha_k = O(\beta_k)$ and $\beta_k = O(\alpha_k)$. The i−th component of a vector u will be denoted by $u_{(i)}$. The unit open (resp. closed) ball centered at 0 will be denoted by B (resp. $\bar{B}$). If A and B are two square symmetric matrices of the same order, we shall write $A < B$ (resp. $A \le B$) to mean that $B-A$ is positive definite (resp. positive semi-definite).

3. The longitudinal displacement

In unconstrained optimization (min $\psi(u)$), quasi-Newton methods locally aim to approximate the Hessian of ψ at a solution u_*. Therefore, the change in the gradient of ψ between two successive iterates u_k and u_{k+1} gives some information on this Hessian and the vectors γ_k and δ_k used in the update formula are usually taken as follows:

$$\gamma_k := \nabla\psi(u_{k+1}) - \nabla\psi(u_k) ,$$

$$\delta_k := u_{k+1} - u_k = \tau_k \, v_k .$$

where τ_k is some step-size in the descent direction v_k of ψ at u_k. Wolfe's step-size selection procedure consists in finding a step-size $\tau = \tau_k$ such that both following inequalities are satisfied:

$$\psi(u_k+\tau v_k) \leq \psi(u_k) + \alpha_1 \, \tau \, \nabla\psi(u_k)^T \, v_k , \tag{3.1}$$

$$\nabla\psi(u_k+\tau v_k)^T \, v_k \geq \alpha_2 \, \nabla\psi(u_k)^T \, v_k , \tag{3.2}$$

where $0 < \alpha_1 < \alpha_2 < 1$. A step-size τ satisfying both inequalities will be called *serious*. Condition (3.1) assures a sufficient decrease of ψ, while condition (3.2) impedes the step-size τ to be too small. A global convergence result can be obtained with these conditions, see Wolfe (1969). An important advantage of this way to select the step-size in the framework of quasi-Newton methods is that condition (3.2) automatically assures the positivity of $\gamma_k^T \, \delta_k$ and as a result the positive definiteness of the updated matrices.

In reduced methods for constrained optimization, an approximation G_k of G_*, the projected Hessian of the Lagrangian, is updated. Here, it is the change in the reduced gradient that gives information on G_*, as suggested by the following formula (see Stoer (1984)):

$$g'(x_*) = (Z(x)^{-T} \, (\nabla f(x) + A(x)^T \, \lambda_*))'(x_*) = Z(x_*)^{-T} \, L_* , \tag{3.3}$$

where we used (1.9) and the second optimality condition in (1.2). Comparing (1.13) and (3.3), we see that G_* is a part of $g'(x_*)$. This is essentially due to the unfeasibility because in this case, any function with value in $\mathbb{R}^{n-m}$ used to obtain information on G_* is defined in $\mathbb{R}^n$ and not on a particular manifold of dimension $n-m$; hence, its Jacobian is a matrix of dimension $(n-m) \times n$ and not of order $n-m$. Therefore an accurate information is obtained on G_* if asymptotically the change in the reduced gradient is given for a displacement along the tangent space $R(Z(x_*)^-)$. This is the basic idea of an update scheme proposed by Coleman and Conn (1984) whose superlinear convergence has been proved by Byrd (1984) and Gilbert (1986,a and 1987). Note that another choice is possible but needs the use of an update criterion: see Nocedal and Overton (1985) for the algorithm of Gabay and Gilbert (1986,a and 1987) for the algorithm of Coleman and Conn. Here we adopt the first strategy: when the unit step-size is accepted (and it will be asymptotically) we take for algorithm (1.6)-(1.7):

$$\gamma_k^1 := g(x_{k+1}) - g(y_k) ,$$

$$\delta_k^1 := Z(y_k)\ t_k = Z(y_k)\ (x_{k+1} - y_k) .$$

The step δ_k^1 is called the ***reduced longitudinal displacement***. Note that the condition $(\gamma_k^1)^T\ \delta_k^1 > 0$ is always satisfied asymptotically because, using (3.3) and supposing that the sequence (y_k) converges to a solution x_* of (1.1) with $t_k \longrightarrow 0$, we have (we use $t_k = Z(y_k)^-\ \delta_k^1$):

$$\gamma_k^1 = Z(x_*)^{-T}\ L_*\ t_k + o(||t_k||) = G_*\ \delta_k^1 + o(||\delta_k^1||)$$

Hence, $(\gamma_k^1)^T\ \delta_k^1$ is positive for k large if G_* is positive definite. However, this condition (1.19) is not necessarily satisfied when y_k is far from x_*, even if a step-size τ is introduced to scale the tangent step t_k:

$$\gamma_k^\tau := g(y_k+\tau t_k) - g(y_k) ,$$

$$\delta_k^\tau := \tau\ Z(y_k)\ t_k .$$

The following counter-example confirms this claim.

Suppose that $n = 2$, $m = 1$, $f(y) := y_{(2)}$, $c(y) := (||y||_2^2 - 1)/2$ ($||\bullet||_2$ is the l_2 norm) and take $\omega = \omega_\beta := \beta B \backslash \beta^{-1}\bar{B}$ with $\beta > 1$. For this data, the unique solution of problem (1.1) is $y_{*(1)} = 0$ and $y_{*(2)} = -1$. We have $A(y) = y^T$. At any point in ω_β, we may use the following orthogonal decomposition of $I\!R^2$:

$$A(y)^- := y/||y||_2^2 ,\quad Z(y)^- := \tilde{y} ,$$

where $\tilde{y}_{(1)} := y_{(2)}$ and $\tilde{y}_{(2)} := -\ y_{(1)}$. To these choices corresponds a unique matrix $Z(y)$ satisfying (1.11) and (1.12): it writes $Z(y) = \tilde{y}^T/||y||_2^2$. The hypothesis of Section 2 are satisfied on ω_β for any $\beta > 1$. We have $g(y) = -\ y_{(1)}$, $t(y) = H\ y_{(1)}\ \tilde{y}$ and $g(y+\tau t(y)) = -\ y_{(1)}(1+\tau H y_{(2)})$. As $\delta^\tau = \tau\ H\ y_{(1)}$, if we suppose $y_{(1)}$ and H positive, the positivity of $(\gamma^\tau)^T\ \delta^\tau$ is equivalent to $g(y+\tau t(y)) > g(y)$, i.e. $-\ \tau\ H\ y_{(1)}\ y_{(2)} > 0$, which is never satisfied for any positive step-size τ when $y_{(2)}$ is also positive.

On the other hand, if we choose a search path $y_k^M(\tau)$ on the manifold $M(y_k) := c^{-1}(c(y_k))$ through $y_k^M(0) := y_k$ (this submanifold of $I\!R^n$ exists because c is a submersion), by seeing the problem of minimizing f on $M(y_k)$ as the one of minimizing $\psi = f\ o\ \xi$ on $I\!R^{n-m}$, where ξ is a parametric representation of $M(y_k)$ around y_k, it is possible to satisfy both Wolfe's conditions (3.1) and (3.2). More precisely, we define a path on $M(y_k)$ by $y_k^M(\tau) := \xi(u_k+\tau v_k)$ ($\xi : U_k \longrightarrow I\!R^n$ is locally defined on the neighborhood U_k of u_k in $I\!R^{n-m}$, and u_k is such that $y_k = \xi(u_k)$) with $v_k := Z(y_k)\ t_k$ and we take $Z(y_k^M(\tau))^- := \xi'(u_k+\tau v_k)$ as the basis of the tangent space to $M(y_k)$ at $y_k^M(\tau)$. Then, $\nabla\psi(u_k+\tau v_k) = g(y_k^M(\tau))$ and v_k is a descent direction of ψ at u_k, since $\nabla\psi(u_k)^T v_k = g(y_k)^T Z(y_k)\, t_k = -\, g(y_k)^T H_k\, g(y_k)$ is negative. Hence, a step-size τ_k such that (3.1)-(3.2) are satisfied at $\tau = \tau_k$ exists if the U_k is sufficiently large and if, for instance, f is bounded from below on $M(y_k)$ (See Wolfe (1969)). Rewriting condition (3.2) at $\tau = \tau_k$ in terms of the reduced gradient (i.e. $g(y_k^M(\tau_k))^T Z(y_k)\, t_k \geq \alpha_2\, g(y_k)^T Z(y_k)\, t_k$) shows that condition (1.19) is satisfied with $\gamma_k = \gamma_k^M$ and δ_k given by

$$\gamma_k^M := g(y_k^M(\tau_k)) - g(y_k)\ ,$$

$$\delta_k := \tau_k\, Z(y_k)\, t_k\ .$$

Without any reference to a parametric representation of $M(y_k)$, the search trajectory $y_k^M(\tau)$ may be defined by an ordinary differential equation (where the dot stands for a derivative according to τ) :

$$\begin{cases} \dot{y}_k^M(\tau) = Z(y_k^M(\tau))^-\, Z(y_k)\, t_k\ , \\ y_k^M(0) = y_k\ , \end{cases} \tag{3.4}$$

while Wolfe's conditions can be rewritten as follows:

$$\theta(y_k^M(\tau)) \leq \theta(y_k) + \alpha_1\, \tau\, \nabla f(y_k)^T\, t_k\ , \tag{3.5}$$

$$g(y_k^M(\tau))^T\, Z(y_k)\, t_k \geq \alpha_2\, g(y_k)^T\, Z(y_k)\, t_k\ , \tag{3.6}$$

where θ is defined in (1.16) with a supposed convex function φ. In (3.5), we have used the function θ instead of the function f and this is licit because $\varphi(c(y_k^M(\tau)))$ does not vary with τ. This small change, however, is important for the sequel because it is indeed the penalty function θ that has to be decreased and not f.

In view of the counter-example and the success of the path $y_k^M(\tau)$, one possible direction of investigation could be to try to build an approximation of the path $y_k^M(\tau)$ using an approximation scheme for the differential equation (3.4). But, on the one hand, this is usually too expansive and, on the other hand, for any τ for which (3.6) would not be satisfied the question of the sharpness of the approximation would arise as a leitmotiv: as shown by the counter-example, the linear approximation $(y_k + \tau t_k)$ is sometimes inadequate, so, what about the current one?

Fortunately, the situation can be sorted out by trying to satisfy both inequalities (3.5) and (3.6) in the following way.

Let us remark first that inequality (3.5) is satisfied for τ small along the linear path $y_k^0(\tau) := y_k + \tau t_k$ $(\tau > 0)$ instead of $y_k^M(\tau)$:

$$\theta(y_k^0(\tau)) \leq \theta(y_k) + \alpha_1\, \tau\, \nabla f(y_k)^T\, t_k\ . \tag{3.7}$$

Indeed, as in (1.17), the right derivative of the left hand side of (3.7) at $\tau = 0$ is then $\nabla f(y_k)^T t_k$, which is negative (we suppose t_k different from zero and H_k positive definite, as usual) and α_1 is less than 1. On the other hand, by continuity and because α_2 is less than 1, the inequality corresponding to (3.6):

$$g(y_k^0(\tau))^T\, Z(y_k)\; t_k \geq \alpha_2\; g(y_k)^T\; Z(y_k)\; t_k\,, \tag{3.8}$$

is not satisfied for small step τ along $y_k^0(\tau)$. Then, we may define $\tau_k^1 :=$ $\sup\,\{\ \tau' > 0 :$ for all τ in $[0\ ,\tau'[\ ,\ y_k^0(\tau)$ is in ω, (3.7) is satisfied and (3.8) is not satisfied $\}$. If $y_k^1 := y_k^0(\tau_k^1)$ is not in ω, we shall consider that the algorithm has failed. Otherwise, (3.7) is satisfied at y_k^1 (by continuity). Then, if (3.8) is satisfied at y_k^1 (which is the only possibility in the unconstrained case), τ_k^1 is a serious step. Otherwise, this means, by continuity, that (3.7) is not satisfied for $\tau > \tau_k^1$ in a neighborhood of τ_k^1. In this last case, $\theta(y_k^1) = \theta(y_k) + \alpha_1\,\tau_k^1\,\nabla f(y_k)^T\,t_k$ and the search to satisfy (3.7) and (3.8) may be pursued from y_k^1 in the direction $Z(y_k^1)^-\;Z(y_k)\;t_k$. To see this, it is enough to remark that, if $y_k^1(\tau)$ is defined by $y_k^1(\tau) := y_k^0(\tau)$ for $0 \leq \tau \leq \tau_k^1$ and $y_k^1(\tau) := y_k^1 + (\tau{-}\tau_k^1)\;Z(y_k^1)^-\;Z(y_k)\;t_k$ for $\tau > \tau_k^1$, the following inequality holds

$$\theta(y_k^1(\tau)) \leq \theta(y_k) + \alpha_1\,\tau\,\nabla f(y_k)^T\;t_k$$

for $\tau > \tau_k^1$ in a neighborhood of τ_k^1. And, this is true because on the contrary we would have a sequence of $\tau > \tau_k^1$, converging to τ_k^1 with

$$\theta(y_k^1(\tau)) - \theta(y_k^1) > \alpha_1\;(\tau{-}\tau_k^1)\;\nabla f(y_k)^T\;t_k\,.$$

Dividing by $(\tau{-}\tau_k^1)$ and taking the limit as τ tends to τ_k^1 would give:

$$(\theta\ o\ y_k^1)'(\tau_k^1\ ;1) \geq \alpha_1\;\nabla f(y_k)^T\;t_k > \alpha_2\;g(y_k)^T\;Z(y_k)\;t_k\,.$$

But

$$\begin{aligned}(\theta\ o\ y_k^1)'(\tau_k^1\ ;1) &= f'(y_k^1)\cdot(Z(y_k^1)^- Z(y_k)t_k) + \varphi'(c(y_k^1)\ ;\ A(y_k^1)Z(y_k^1)^- Z(y_k)t_k)\\ &= g(y_k^1)^T\;Z(y_k)\;t_k\,,\end{aligned}$$

because of (1.9). So $g(y_k^1)^T\;Z(y_k)\;t_k\; > \alpha_2\;g(y_k)^T\;Z(y_k)\;t_k$. A contradiction, because (3.8) was supposed to be unsatisfied at $y_k^1 = y_k^0(\tau_k^1)$. Now, we can continue and define $\tau_k^2 := \ \sup\,\{\ \tau' > \tau_k^1$: for all τ in $[\tau_k^1\ ,\tau'[\ ,\ y_k^1(\tau)$ is in ω, $\theta(y_k^1(\tau)) \leq$ $\theta(y_k) + \alpha_1\,\tau\,\nabla f(y_k)^T\;t_k\ ,\ g(y_k^1(\tau))^T\;Z(y_k)\;t_k \geq \alpha_2\;g(y_k)^T\;Z(y_k)\;t_k\ \}$, $y_k^2 := y_k^1(\tau_k^2)$ and so on. Therefore, the search can be pursued along piecewise linear path, as long as a serious step-size is not met.

To obtain an implementable version of this algorithm, two questions, which constitute our program up to the end of this section, have to be clarified:

(1) the values τ_k^i of the step-length at which the search is reoriented is not attainable by calculation and should be redefined,

(2) the algorithm should be shown to terminate in a finite number of iterations.

The last question will be the subject of Theorem 3.3, while for the first question, we may refer to what is done in unconstrained optimization to find a serious step-size in the sense of Wolfe. Indeed, in this case as well, if the step-size corresponding to our τ_k^1 solves the problem, it is never calculated but only approximated and this is possible because it must exist a left neighborhood of it constituted of serious step-sizes. For example, Lemaréchal (1981) has proposed an algorithm to find a serious step-size in unconstrained optimization. Let us recall it here in terms of the function ψ introduced at the beginning of the Section.

Lemaréchal's algorithm: (3.9)

1. set $\underline{\tau} := 0$, $\overline{\tau} := \infty$; choose $\tau > 0$
2. repeat:
   - 2.1. if (3.1) is not satisfied
   - 2.2. then { $\overline{\tau} := \tau$; $\tau :=$ INTERPOL $(\underline{\tau}$, $\overline{\tau})$ }
   - 2.3. else if (3.2) is satisfied
   - 2.4. then go out // τ is serious
   - 2.5. else { $\underline{\tau} := \tau$;
   - 2.6. if $\overline{\tau} = \infty$
   - 2.7 then $\tau :=$ EXTRAPOL $(\underline{\tau})$
   - 2.8 else $\tau :=$ INTERPOL $(\underline{\tau}$, $\overline{\tau})$ }

So, the algorithm tries to trap a step-size like τ_k^1 in an interval $[\underline{\tau}$, $\overline{\tau}]$. The step-size $\overline{\tau}$ is said to be *too large* because it does not satisfy (3.1), hence some step-size like τ_k^1 must exist in $[0$, $\overline{\tau}]$. The step-size $\underline{\tau}$ is said to be *too small* because it is less than $\overline{\tau}$ and satisfies (3.1) but not (3.2), hence some step-size like τ_k^1 must exist in $[\underline{\tau}$, $\overline{\tau}]$. The algorithm uses two functions: INTERPOL gives a step-size τ between the two finite values $\underline{\tau}$ and $\overline{\tau}$ and EXTRAPOL gives a step-size τ greater than $\underline{\tau}$. Some conditions on these functions are required in order to assure the global convergence of the algorithm.

We shall adapt this algorithm to our situation by modifying the direction of search each time a step-size is recognized as too small. These step-sizes will constitute our new τ_k^i's. Note that this change in the direction of search at a point y_k^i is free of charge because an inequality like (3.6) or (3.8) has to be tested at y_k^i and therefore the new basis $Z(y_k^i)^-$ is available and the new search direction $Z(y_k^i)^- \, Z(y_k) \, t_k$, as well.

Before stating our algorithm we need to define precisely the search path and to give the properties required for the interpolation and extrapolation functions.

Being given l positive numbers:

$$0 =: \tau_k^0 < \tau_k^1 < \cdots < \tau_k^l , \tag{3.10}$$

we define by induction the points y_k^i and the piecewise linear trajectories $y_k^i(\tau)$, $\tau \geq 0$ for $0 \leq i \leq l$:

$$y_k^0 := y_k ,$$

$$y_k^0(\tau) := y_k + \tau\, t_k = y_k + \tau\, Z(y_k)^-\, Z(y_k)\, t_k \quad \text{for} \quad \tau \geq 0 \tag{3.11}$$

and for $1 \leq i \leq l$,

$$y_k^i := y_k^{i-1}(\tau_k^i) , \tag{3.12}$$

$$y_k^i(\tau) := \begin{cases} y_k^{i-1}(\tau) & \text{for} \quad 0 \leq \tau \leq \tau_k^i , \\ y_k^i + (\tau - \tau_k^i)\, Z(y_k^i)^-\, Z(y_k)\, t_k & \text{for} \quad \tau > \tau_k^i . \end{cases} \tag{3.13}$$

Therefore, if the dot stands for a right derivative, we have

$$\dot{y}_k^l(\tau_k^i) = Z(y_k^i)^-\, Z(y_k)\, t_k\, , \quad 0 \leq i \leq l . \tag{3.14}$$

So, the path $y_k^l(\tau)$ may be seen as an Euler approximation of the solution $y_k^M(\tau)$ of (3.4) on $[0\ ,\ \tau_k^l]$ for the discretization (3.10) in τ.

Lemma 3.1. *If $(\tau_k^i)_{i \geq 1}$ is an increasing sequence of positive numbers that converges to some $\bar{\tau}_k$ and if $(y_k^i)_{i \geq 1}$ defined in (3.12)-(3.13) remains in ω, then $(y_k^i)_{i \geq 1}$ converges in $\mathbb{R}^n$.*

Proof. We have

$$y_k^i = y_k + \sum_{j=0}^{i-1} (\tau_k^{j+1} - \tau_k^j)\, Z(y_k^j)^-\, Z(y_k)\, t_k .$$

Because $y_k^j \in \omega$, $(Z(y_k^j)^-)_{j \geq 0}$ is bounded. Therefore, the sum in the right hand side is absolutely convergent, hence converges. And so does (y_k^i) . □

The generalization of Wolfe's criteria then writes: find l positive numbers $(\tau_k^i)_{1 \leq i \leq l}$ verifying (3.10) and a $\tau > \tau_k^l$ such that

$$\theta(y_k^l(\tau)) \leq \theta(y_k) + \alpha_1\, \tau\, \nabla f(y_k)^T\, t_k\, , \tag{3.15}$$

$$g(y_k^l(\tau))^T\, Z(y_k)\, t_k \geq \alpha_2\, g(y_k)^T\, Z(y_k)\, t_k\, , \tag{3.16}$$

where $0 < \alpha_1 < \alpha_2 < 1$ are given. We shall need an *interpolation function* $J : \omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+ := [0\ ,\ \infty[$ such that:

$$(y,\tau) \longrightarrow J(y,\tau) \quad \text{is continuous on } \omega \times I\!R_+ \ , \tag{3.17}$$

$$J(y,\tau) \in \,]0 \ , \tau[\quad \text{for all } y \text{ in } \omega \text{ and all } \tau > 0 \ , \tag{3.18}$$

$$J_y^p(\tau) := (J_y \ o \ .. \ (p \text{ times}) \ .. \ o \ J_y)(\tau) \longrightarrow 0 \ \text{ as } \ p \longrightarrow \infty \ , \tag{3.19}$$

where $J_y(\tau) := J(y,\tau)$. From (3.17) and (3.18), we deduce that for y in ω and τ non negative, $J(y,\tau) = 0$ if and only if $\tau = 0$. We shall also need an *extrapolation function* $E : \omega \longrightarrow I\!R_+$ such that:

$$y \longrightarrow E(y) \text{ is continuous on } \omega \ , \tag{3.20}$$

$$E(y) > 0 \text{ for all } y \text{ in } \omega \ . \tag{3.21}$$

Lemma 3.2. *If (y_j) is a converging sequence in ω, (τ_j) is a bounded sequence of positive numbers such that $(J(y_j,\tau_j))$ converges to zero, then (τ_j) converges to zero.*

Proof. Let y in ω be the limit point of (y_j) and $(\tau_{j'})$ be a subsequence of (τ_j) that converges to some τ. Then, by (3.17), $J(y_{j'},\tau_{j'}) \longrightarrow J(y,\tau) = 0$, hence $\tau = 0$ and all the sequence (τ_j) converges to zero. □

We can now state the

Longitudinal search algorithm: (3.22)

1. if $t_k = 0$ then go out
2. set $l := 0$, $\tau_k^0 := 0$; choose $\tau > 0$
3. repeat:
   - 3.1. if $y_k^l(\tau)$ is not in ω or (3.15) is not satisfied
   - 3.2. then $\tau := \tau_k^l + J(y_k^l,\tau-\tau_k^l)$
   - 3.3. else { if (3.16) is satisfied
   - 3.4. then { $\tau_k := \tau$; go to statement 4 } // τ is serious
   - 3.5. else { $l := l+1$; $\tau_k^l := \tau$; $\tau := \tau_k^l + E(y_k^l)$ } }
4. $l_k := l$

In statement 2, the choice $\tau = 1$ is recommended if the algorithm is used within the context of secant methods because in this case the unit step-size is essential to obtain the superlinear convergence.

We have added in statement 3.1 another reason to decrease τ: the points y_k^l must belong to ω. Therefore a serious step-size may not be found because ω is too small and the algorithm may loop in statement 3. This is one of both reasons for looping (see the theorem below).

Let us remark that the algorithm will not cycle between statment 3.1 and 3.2 because otherwise τ would decrease to τ_k^l by hypothesis (3.19); but, $y_k^l := y_k^l(\tau_k^l)$ is in ω by construction so $y_k^l(\tau)$ is also in ω for τ close to τ_k^l and , on the other hand, inequality (3.15) is satisfied for τ close to τ_k^l. Therefore the test 3.1 is always rejected after a finite number of loops 3.1-3.2. Consequently, if the algorithm loops in statment 3, a sequence $(y_k^l)_{l\geq 0}$ is built in ω.

We now give the main result of this section, which shows that, apart from some pathological situations, a serious step-size is found in a finite number of iterations.

Theorem 3.3. *Let θ be the function defined on ω by (1.16) with φ convex and continuous on a neighborhood of $c(\omega)$. Let y_k be a point in ω such that $g(y_k) \neq 0$. Let H_k be a symmetric positive definite matrix of order $n-m$. Then, if the longitudinal search algorithm (3.22) with the definitions (3.10)-(3.13) and the hypotheses (3.17)-(3.21) is applied from y_k, one of the following situations occurs:*

(i) the algorithm terminates in a finite number l_k of loops 3.1-3.5, with a point $x_{k+1} := y_k^{l_k}(\tau_k)$ satisfying both inequalities (3.15) and (3.16) with $l = l_k$ and $\tau = \tau_k$,

(ii) the algorithm builds a sequence $(y_k^l)_{l\geq 0}$ in ω and either $(\theta(y_k^l))_{l\geq 0}$ tends to $-\infty$ or $(y_k^l)_{l\geq 0}$ converges to a point on the boundary of the open set ω.

Proof. Clearly, by the remark preceding the theorem, if a finite number l_k of points y_k^l are calculated, this means that a step-size τ_k has been found in statement 3.4 and that $y_k^{l_k}(\tau_k)$ satisfies both (3.15) and (3.16). So let us suppose the contrary, i.e. that a sequence $(y_k^l)_{l\geq 0}$ is built in ω. Let us suppose also that the sequence $(\theta(y_k^l))_{l\geq 0}$ is bounded from below and that $(y_k^l)_{l\geq 0}$ does not converge to a point on the boundary of ω. We have to prove that these suppositions yield to a contradiction.

For all l, we have by construction:

$$\theta(y_k^l) \leq \theta(y_k) + \alpha_1\, \tau_k^l\, \nabla f(y_k)^T\, t_k\,, \tag{3.23}$$

$$g(y_k^l)^T\, Z(y_k)\, t_k < \alpha_2\, g(y_k)^T\, Z(y_k)\, t_k\,. \tag{3.24}$$

Because $\nabla f(y_k)^T\, t_k = -\, g(y_k)^T\, H_k\, g(y_k)$ is negative and $(\theta(y_k^l))_{l\geq 0}$ is bounded from below, (3.23) shows that $(\tau_k^l)_{l\geq 0}$ is bounded. As τ_k^l increases with l, the sequence converges to some $\bar{\tau}_k$ and by Lemma 3.1, $(y_k^l)_{l\geq 0}$ converges to some $\bar{y}_k$ in $\mathbb{R}^n$. According to the suppositions, $\bar{y}_k$ is in ω. Let us show that

$$\theta(\bar{y}_k) = \theta(y_k) + \alpha_1\, \bar{\tau}_k\, \nabla f(y_k)^T\, t_k\,. \tag{3.25}$$

In view of (3.23) and by continuity, it is enough to prove that the left hand side of (3.25) is not less than the right hand side. For this, let us remark that there exists an integer l^0 such that for $l \geq l^0$, we have for some $\bar{\tau}_k^l > \tau_k^{l+1}$:

$$\tau_k^{l+1} = \tau_k^l + J(y_k^l, \bar{\tau}_k^l - \tau_k^l) \ . \qquad (3.26)$$

Indeed, on the contrary, we would have $\tau_k^{l+1} = \tau_k^l + E(y_k^l)$ for some subsequence of l's and at the limit on those l's, we would have, by (3.20), $E(\bar{y}_k) = 0$, which is in contradiction with hypothesis (3.21). Now, by construction, $\bar{\tau}_k^l \leq \tau_k^l + E(y_k^l)$. So $(\bar{\tau}_k^l)_{l \geq l^0}$ is also bounded. Then, the limit in (3.26) and Lemma 3.2 show that $\bar{\tau}_k^l$ converges to $\bar{\tau}_k$. The equality (3.26) also means that (3.15) is not satisfied at $y_k^l(\bar{\tau}_k^l)$, i.e.

$$\theta(y_k^l(\bar{\tau}_k^l)) > \theta(y_k) + \alpha_1 \, \bar{\tau}_k^l \, \nabla f(y_k)^T \, t_k \ .$$

Because $y_k^l(\bar{\tau}_k^l) = y_k^l + (\bar{\tau}_k^l - \tau_k^l) \, Z(y_k^l)^- \, Z(y_k) \, t_k$ converges to $\bar{y}_k$, the equality (3.25) is proved by taking the limit on l in this last inequity.

Taking the limit on l in (3.24) and noting that $0 < \alpha_1 < \alpha_2$, we see that there will be a contradiction (and therefore we shall have proved the theorem) if we show that

$$g(\bar{y}_k)^T \, Z(y_k) \, t_k \geq \alpha_1 \, g(y_k)^T \, Z(y_k) \, t_k \ . \qquad (3.27)$$

For this, we build a sequence of points $(z_k^p)_{p \geq 0}$ in ω of the form

$$z_k^p = \bar{y}_k + \eta^p \, Z(\bar{y}_k)^- \, Z(y_k) \, t_k \ , \qquad (3.28)$$

and verifying

$$\theta(z_k^p) \geq \theta(y_k) + \alpha_1 \, (\bar{\tau}_k + \eta^p) \, g(y_k)^T \, Z(y_k) \, t_k \ , \qquad (3.29)$$

where $(\eta^p)_{p \geq 0}$ is a sequence of positive numbers converging to zero. Therefore, using (3.25), we get

$$\frac{\theta(z_k^p) - \theta(\bar{y}_k)}{\eta^p} \geq \alpha_1 \, g(y_k)^T \, Z(y_k) \, t_k \ .$$

Hence, inequality (3.27), by taking the limit on p in this inequality.

The sequence $(z_k^p)_{p \geq 0}$ is built by induction and we begin with z_p^0. We have already seen in getting (3.26) that we may find a positive integer l^0 such that for $l \geq l^0$, (3.15) is not satisfied at $y_k^l(\tau_k^l + E(y_k^l))$. Therefore, for $l \geq l^0$,

$$\theta(y_k^l(\tau_k^l + E(y_k^l))) > \theta(y_k) + \alpha_1 \, (\tau_k^l + E(y_k^l)) \, g(y_k)^T \, Z(y_k) \, t_k \ . \qquad (3.30)$$

If we set

$$z_k^0 := \lim_{l \longrightarrow \infty} y_k^l(\tau_k^l + E(y_k^l)) = \bar{y}_k + E(\bar{y}_k) \, Z(\bar{y}_k)^- \, Z(y_k) \, t_k \ ,$$

and $\eta^0 := E(\bar{y}_k)$, we obtain (3.28) and (3.29) for $p = 0$. Similarly, for $p \geq 1$, we may find a positive integer $l^p \geq l^{p-1}$ (defined by induction) such that for all $l \geq l^p$, (3.15) is not satisfied at

$$z_k^{l,p} := y_k^l + \eta^{l,p}\ Z(y_k^l)^-\ Z(y_k)\ t_k\ ,$$

where

$$\eta^{l,p} := J_{y_k^l}^p(E(y_k^l)).$$

Indeed, otherwise, l^p being greater than l^{p-1}, we would have for a subsequence of l's:

$$\tau_k^{l+1} = \tau_k^l + J_{y_k^l}^p(E(y_k^l))$$

and the limit on l, would give

$$(J_{\bar{y}_k}\ o\ ..\ (p\ \text{times})\ ..\ o\ J_{\bar{y}_k})(E(\bar{y}_k)) = 0\ .$$

Which would imply $E(\bar{y}_k) = 0$, in contradiction with (3.21). Therefore, for $l \geq l^p$, we have

$$\theta(z_k^{l,p}) > \theta(y_k) + \alpha_1\ (\tau_k^l + \eta^{l,p})\ g(y_k)^T\ Z(y_k)\ t_k\ .$$

As $(z_k^{l,p})_{l\geq 0}$ converges to z_k^p given by (3.28) with

$$\eta^p := J_{\bar{y}_k}^p(E(\bar{y}_k))\ ,$$

the limit in the last inequality gives (3.29). Moreover, $(\eta^p)_{p\geq 0}$ converges to zero because of property (3.19). □

So, being given a point y_k satisfying the hypotheses of Theorem 3.3, the longitudinal search will usually give a point

$$x_{k+1} := y_k^{l_k}(\tau_k) \tag{3.31}$$

satisfying both inequalities (3.15) and (3.16). We shall need further, the following inequality.

Proposition 3.4. *With the hypotheses of Theorem 3.3, if starting from a point y_k in ω, the longitudinal search algorithm gives a point x_{k+1} in ω and a step-size τ_k, we have*

$$||x_{k+1} - y_k|| \leq C\ \tau_k\ ||t_k||\ , \tag{3.32}$$

where C is a positive constant that only depends on $c(.)$, $Z(.)$ and $Z(.)^-$ on ω.

Proof. We have

$$x_{k+1}-y_k = \sum_{i=0}^{l_k-1} (\tau_k^{i+1}-\tau_k^i)\ Z(y_k^i)^-\ Z(y_k)\ t_k\ ,$$

from which (3.32) follows. □

4. The algorithm

In secant methods, it is commonly considered that a globalizing technique is successful if the unit step-size is asymptotically accepted by the search algorithm because then, the superlinear convergence of the local method is not prevented from occurring. In the case of the longitudinal search algorithm (3.22), this means that $l_k = 0$ and $\tau_k = 1$ should be accepted after a finite number of iterations. In fact, this depends on three factors: the search direction t_k, i.e. the matrix H_k, the penalty function θ and the constants α_1 and α_2 in (3.15)-(3.16).

Because G_k is updated to be a good approximation of the projected Hessian of the Lagrangian, which is a condition imposed by the local analysis, the point (y_k+t_k) will be asymptotically a good approximation of a minimizer of the Lagrangian in the tangent plan $y_k+R(Z(y_k)^-)$. Note, indeed, that if $G_k = Z(y_k)^{-T}\ L(y_k,\lambda)\ Z(y_k)^-$, we have $t_k :=$ arg min $\{\ l(y_k,\lambda) + \nabla_x l(y_k,\lambda)^T t + t^T L(y_k,\lambda)t/2 : t \in R(Z(y_k)^-)\ \}$, for any λ. Therefore, the unit step-size has some chance to be accepted if θ is close to the Lagrangian. Finally, the condition on the α_i's will be simply, $\alpha_1 < 1/2$ because then conditions (3.15) and (3.16) accept the minimum of a quadratic function.

We choose as penalty function the nondifferentiable augmented Lagrangian:

$$l_p(x,\mu) := f(x) + \mu^T\ c(x) + p\ ||c(x)||\ , \tag{4.1}$$

where $||\bullet||$ is a norm on $\mathbb{R}^m$. This penalty function is exact, i.e. that a solution x_* of problem (1.1) is a local minimizer (here strict) of $l_p(\bullet,\mu)$, if $p > ||\mu-\lambda_*||_D$ where λ_* is the multiplier associated to x_* and $||\bullet||_D$ is the dual norm of $||\bullet||$ on $\mathbb{R}^m$. This result may be derived as a variant of a result of Han and Mangasarian (1979) by considering the problem min $\{\ f(x) + \mu^T\ c(x) : x \in \omega\ ,\ c(x) = 0\ \}$, which is equivalent to problem (1.1) or it may be directly obtained like in Bonnans (1984) where the penalty function (4.1) has been used in connection with the SQP method to obtain the admissibility of the unit step-size.

Another possibility could have been to use the differentiable augmented Lagrangian obtained by replacing in (4.1), $p||c(x)||$ by $(p/2)||c(x)||_2^2$ (where $||\bullet||_2$ is the l_2 norm), which is exact if p is greater than some positive threshold $\underline{p}$. The advantage of l_p in (4.1) is that the threshold of p is easy to calculate. This is important, because, as we have seen, we shall need to make l_p close to the

Lagrangian function so that the unit step-size will be accepted. With the penalty function (4.1), this will be done simply by improving the estimate μ of λ_* as the iterates progress to a solution x_* and by decreasing p if necessary and if the requirement $p > ||\mu-\lambda_*||_D$ allows it.

The path from y_k to x_{k+1}, given in (3.31), may be obtained by using $l_p(\bullet,\mu)$ as penalty function in algorithm (3.22). So, it remains to bring out conditions for the feasibility of a linear search on l_p starting at x_{k+1} in the direction r_{k+1}, or at x_k in the direction r_k. The directional derivation in x of $l_p(\bullet,\mu)$ at x_k in the direction $r_k := -\ A(x_k)^-\ c(x_k)$ writes:

$$l_p'(x_k,\mu\ ;\ r_k) = (\lambda(x_k)-\mu)^T\ c(x_k)\ -\ p\ ||c(x_k)||\ , \tag{4.2}$$

where we used the multiplier estimate $\lambda(x)$ given in (1.15). Therefore, r_k is a descent direction of $l_p(\bullet,\mu)$ at x_k, if $p > ||\lambda(x_k)-\mu||_D$. This shows that p will have to be adapted sometimes in order to preserve this inequality before doing the transversal step. We shall denote by p_k the value of the penalty parameter at iteration k. In the same way, we shall see that μ will have to be modified at some iteration and we shall denote by μ_k its value at iteration k. Therefore, a condition to satisfy at each iteration (from x_k to x_{k+1}) is:

$$p_k \geq ||\lambda(x_k)-\mu_k||_D + \underline{p}_k\ , \tag{4.3}$$

where $\underline{p}_k$ is some positive number.

Let ρ_k denote the step-size along the *transversal displacement* r_k:

$$y_k := x_k + \rho_k\ r_k\ . \tag{4.4}$$

We shall determine ρ_k by Armijo's procedure (see Armijo (1966)). We choose β in $]0\ ,\ 1[$ and we take ρ_k in the form

$$\rho_k := \beta^{b_k}\ , \tag{4.5}$$

where b_k is the smallest non negative integer such that

$$x_k + \rho_k\ r_k \in \omega$$

and

$$\begin{aligned} &l_{p_k}(x_k+\rho_k r_k,\mu_k) \\ &\qquad \leq l_{p_k}(x_k,\mu_k) + \alpha\,\rho_k\,[(\lambda(x_k)-\mu_k)^T\ c(x_k) - p_k\ ||c(x_k)||]\,, \end{aligned} \tag{4.6}$$

where α is a given constant in $]0\ ,\ 1[$. So, if (4.3) is satisfied and x_k is in ω (an open set), such a b_k always exists.

We can now outline our reduced secant algorithm.

Algorithm RSA: (4.7)

1. Let be given the constants: $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \alpha_1 < 1/2$, $\alpha_1 < \alpha_2 < 1$, $0 < \epsilon$, $1 < a_i$ $(i = 1,2,3)$.
2. Let x_0 be a point in ω and H_0 be a symmetric positive definite matrix of order $n-m$.
3. Calculate $\lambda(x_0)$ by (1.15), choose $\underline{p}_0 > 0$, set $\mu_0 := \lambda(x_0)$ and $p_0 := S(\underline{p}_0)$ and set the indices $k := 0$ (iterations), $i := 0$ (adaptation of $\underline{p}_k$), $j := 0$ (adaptation of p_k and μ_k).
4. Select a transversal step-length ρ_k by Armijo's procedure (4.5)-(4.6) and set $y_k := x_k + \rho_k\, r_k$.
5. Execute the longitudinal search algorithm (3.22), starting with $\tau = 1$ and using the penalty function $l_{p_k}(\bullet,\mu_k)$ instead of $\theta(\bullet)$ in (3.15) to determine a step-length τ_k and the point x_{k+1} given by (3.31), if possible.
6. Calculate $\epsilon_k := ||g(y_k)|| + ||c(x_{k+1})||$. If $\epsilon_k < \epsilon$ then stop.
7. Update H_k : $\gamma_k := g(x_{k+1})-g(y_k)$, $\delta_k := \tau_k\, Z(y_k)\, t_k$, $H_{k+1} := \overline{BFGS}(H_k,\gamma_k,\delta_k)$.
8. Adapt $\underline{p}_k \longrightarrow \underline{p}_{k+1}$.
9. Adapt $p_k \longrightarrow p_{k+1}$ and $\mu_k \longrightarrow \mu_{k+1}$.
10. Set $k := k+1$ and go to statement 4.

The algorithm calls for some comments. In statement 1, ϵ is a positive convergence threshold and is used in statement 6. The positive constants a_1, a_2 and a_3 will be used in the adaptation rules of $\underline{p}_k$, p_k and μ_k (statement 8 and 9) given further. In statement 2, H_0 can be chosen as the identity matrix but this does not take into account the scaling of the problem. Therefore, a possible choice is to take $H_0 = I$ in the first longitudinal search (statement 5) and to calculate H_1 by updating $h_0\, I$ rather than I, where

$$h_0 := \frac{\gamma_0^T\, \delta_0}{\gamma_0^T\, \gamma_0}$$

is the scalar minimizer of $||h\gamma_0-\delta_0||_2^2$ (see Shanno and Phua (1978)). In statement 3, $\underline{p}_0$ should be taken large enough and the function $S :]0\ ,\ \infty[\ \longrightarrow\ ]0\ ,\ \infty[$ is supposed to satisfy the following properties:

$$S \text{ is non decreasing on }]0\ ,\ \infty[\text{ and } S(a) \geq a \text{ for } a \text{ in }]0\ ,\ \infty[\ , \tag{4.8}$$

$$\text{for all } \underline{a} \leq \bar{a} \text{ in }]0\ ,\ \infty[\ ,\ S([\underline{a}\ ,\ \bar{a}]) \text{ is finite}\ , \tag{4.9}$$

$$S(a) \longrightarrow 0 \text{ as } a \longrightarrow 0+\ . \tag{4.10}$$

This function S will be used again in the adaptation rules of $\underline{p}_k$, p_k and μ_k and these properties will be useful to prove the global convergence of the algorithm. For example, we may follow Bonnans (1984) by taking $S(a) := \min\{ 10^q : a \le 10^q,\ q \text{ integer} \}$. Statement 4 will always succeed because inequality (4.3) will be guaranteed by the adaptation rule of $\underline{p}_k$, p_k and μ_k (statements 8 and 9) and because if statement 5 succeeds, the point x_k is in ω. Note that if $r_k = 0$, $\rho_k = 1$ is always accepted in statement 4! On the other hand, statement 5 may not succeed because one of the situations of (ii) in Theorem 3.3 occurs. In order not to prevent the superlinear convergence from occurring we suppose that the initial τ in the longitudinal search algorithm is chosen equal to 1. In statement 7, the inverse BFGS formula (1.18) is always well defined because, by construction, $\gamma_k^T \delta_k$ is positive.

Before stating the adaptation rules for $\underline{p}_k$, p_k and μ_k in statements 8 and 9, we need to examine in what conditions the unit step-sizes ρ_k and τ_k are accepted in both the transversal and longitudinal displacements. These are the contents of the following two propositions.

Proposition 4.1. *Suppose that algorithm RSA (4.7) produces bounded sequences (μ_k) and (p_k) and a sequence (x_k) in ω that converges in ω to a solution x_* of problem (1.1). Then, with $\Theta_k := (\lambda(x_k)-\mu_k)^T\, c(x_k) - p_k\, ||c(x_k)||$, we have for k large*

$$l_{p_k}(x_k+r_k,\mu_k) - l_{p_k}(x_k,\mu_k) - \alpha\,\Theta_k = (1-\alpha)\,\Theta_k + o(||r_k||)\,. \tag{4.11}$$

Proof. For k large and θ in $[0\,,\,1]$, $x_k+\theta r_k$ is in ω. Then, Taylor's expansions give

$$f(x_k+r_k) = f(x_k) + \lambda(x_k)^T\, c(x_k) + o(||r_k||)$$

and $c(x_k+r_k) = o(||r_k||)$. Consequently, (p_k) and (μ_k) being bounded, we get (4.11). □

Proposition 4.2. *Suppose that algorithm RSA (4.7) produces a bounded sequence (p_k), a sequence (x_k) in ω that converges in ω to a solution x_* of problem (1.1), a sequence (μ_k) that converges to the associated multiplier λ_* and a sequence of non singular reduced matrices (G_k) such that (G_k^{-1}) is bounded and*

$$(G_k - G_*)\, Z(y_k)\, t_k = o(||t_k||)\,. \tag{4.12}$$

Then, with $\Lambda_k := \nabla f(y_k)^T\, t_k$, we have for k large

$$\begin{aligned} &l_{p_k}(y_k+t_k,\mu_k) - l_{p_k}(y_k,\mu_k) - \alpha_1\,\Lambda_k \\ &\qquad \le (\tfrac{1}{2}-\alpha_1)\,\Lambda_k + p_k\, O(||t_k||^2) + o(||t_k||^2)\,, \end{aligned} \tag{4.13}$$

$$\alpha_2\, g(y_k)^T\, Z(y_k)\, t_k - g(y_k+t_k)^T Z(y_k)\, t_k = \alpha_2\, \Lambda_k + o(||t_k||^2)\,. \qquad (4.14)$$

Proof. The sequence (y_k) converges to x_* and because (H_k) is bounded, $y_k+\theta t_k$ is in ω for k large and θ in $[0\,,\,1]$. Then Taylor's expansions give:

$$f(y_k+t_k) = f(y_k) + \Lambda_k + \frac{1}{2}\, f''(x_*) \cdot t_k^2 + o(||t_k||^2)\,,$$

$$c(y_k+t_k) = c(y_k) + \frac{1}{2}\, c''(x_*) \cdot t_k^2 + o(||t_k||^2)\,.$$

Hence, using $\mu_k \longrightarrow \lambda_*$ and the boundedness of (p_k), we get

$$l_{p_k}(y_k+t_k,\mu_k) \le l_{p_k}(y_k,\mu_k) + \Lambda_k + \frac{1}{2}\, t_k^T L_*\, t_k + p_k\, O(||t_k||^2) + o(||t_k||^2)\,.$$

But $t_k = Z(y_k)^-\, Z(y_k)\, t_k = Z(x_*)^-\, Z(y_k)\, t_k + o(||t_k||)$ and $\Lambda_k = -\, g(y_k)^T\, H_k\, g(y_k) = -\, t_k^T\, Z(y_k)^T\, G_k\, Z(y_k)\, t_k$. So, the last inequality becomes

$$\begin{aligned} & l_{p_k}(y_k+t_k,\mu_k) - l_{p_k}(y_k,\mu_k) - \alpha_1\, \Lambda_k \\ & \quad \le (\tfrac{1}{2}-\alpha_1)\, \Lambda_k - \frac{1}{2}\, t_k^T\, Z(y_k)^T\, (G_k-G_*)\, Z(y_k)\, t_k \\ & \quad\quad + p_k\, O(||t_k||^2) + o(||t_k||^2)\,. \end{aligned}$$

From this inequality and from (4.12), we deduce (4.13). On the other hand, by a Taylor's expansion and (3.3), we get

$$\begin{aligned} g(y_k+t_k) &= g(y_k) + Z(x_*)^{-T}\, L_*\, t_k + o(||t_k||) \\ &= g(y_k) + G_*\, Z(y_k)\, t_k + o(||t_k||)\,. \end{aligned}$$

Hence, using again $\Lambda_k = -\, t_k^T\, Z(y_k)^T\, G_k\, Z(y_k)\, t_k$ and (4.12), we obtain

$$\begin{aligned} & \alpha_2\, g(y_k)^T\, Z(y_k)\, t_k - g(y_k+t_k)^T\, Z(y_k)\, t_k \\ & \quad = \alpha_2\, \Lambda_k - \Lambda_k - t_k^T\, Z(y_k)^T\, G_*\, Z(y_k)\, t_k + o(||t_k||^2) \\ & \quad = \alpha_2\, \Lambda_k + t_k^T\, Z(y_k)^T\, (G_k-G_*)\, Z(y_k)\, t_k + o(||t_k||^2) \\ & \quad = \alpha_2\, \Lambda_k + o(||t_k||^2)\,, \end{aligned}$$

which is (4.14). □

Propositions 4.1 and 4.2 give conditions to have the admissibility of the unit step-sizes ρ_k and τ_k, i.e. to have the left hand side of (4.11), (4.13) and (4.14) non positive. This will guide us in the design of the adaptation rules for $\underline{p}_k$, p_k and μ_k.

Inequality (4.11) shows that $0 < \alpha < 1$, inequality (4.3) with $(\underline{p}_k)$ bounded away from zero is sufficient to guarantee $\rho_k = 1$ asymptotically. Indeed, in this case, $\Theta_k \leq - \underline{p}_k \, ||c(x_k)|| \leq - \, C \, ||r_k||$, where C is a positive constant, and the left hand side of (4.11) becomes negative eventually.

By proposition 4.2, we see that μ_k has to be adapted infinitely often so that $\mu_k \longrightarrow \lambda_*$. Condition (4.12) is a sufficient (but not necessary) condition of superlinear convergence for (x_k) (see Byrd (1984), Gilbert (1986,a,c)) and is usually satisfied in practice (see Coleman and Conn (1984) and Gilbert (1987)). Because $\Lambda_k = - \, t_k^T \, Z(y_k)^T \, G_k \, Z(y_k) \, t_k$, the left hand side of (4.14) will be negative asymptotically if (G_k) is positive definite, (G_k^{-1}) is bounded and $\alpha_2 > 0$, while the left hand side of (4.13) will be negative asymptotically if (G_k) is positive definite, (G_k^{-1}) is bounded, $\alpha_1 < 1/2$ and p_k is sufficiently small. But, because p_k has to satisfy the inequality (4.3), this means that $\underline{p}_k$ must be small enough, although non zero as we have just seen.

On the other hand, by modifying μ_k and p_k, we change the merit function at each iteration, which can prevent convergence. So, we have to proceed with caution, and like in Bonnans (1984) we shall not change μ_k and p_k if convergence does not seem to occur. We actually think that here lies Achille's tendon of the algorithm and that some improvement could be brought. For the test of convergence we shall use

$$\epsilon_k^0 := \min \{ \, \epsilon_i : 0 \leq i \leq k \, \} \tag{4.15}$$

We can now precise statements 8 and 9 of algorithm RSA.

Adaptation of $\underline{p}_k$ (statement 8 of algorithm RSA): (4.16)

if $\epsilon_k^0 \leq \epsilon_i^0 / a_1$ and $(l_k \neq 0$ or $\tau_k \neq 1)$

then $\{ \, i := k \, ; \, \underline{p}_{k+1} := \underline{p}_k / a_2 \, \}$ (4.17)

else $\underline{p}_{k+1} := \underline{p}_k$ (4.18)

Adaptation of p_k and μ_k (statement 9 of algorithm RSA): (4.19)

if $\epsilon_k^0 \leq \epsilon_j^0 \, / a_3$

then $\{ \, j := k \, ; \, \mu_{k+1} := \lambda(x_{k+1}) \, ; \, p_{k+1} := S(\underline{p}_{k+1}) \, \}$ (4.20)

else $\{ \, \mu_{k+1} := \mu_k \, ; \, p_{k+1} := \max \, (p_k \, , \, S(||\lambda(x_{k+1}) - \mu_{k+1}||_D + \underline{p}_{k+1}) \, \}$ (4.21)

We recall that the properties of function S have been given in (4.8)-(4.10). We now prove the global convergence of algorithm RSA.

Theorem 4.3. *Suppose that algorithm RSA (4.7) with the adaptation rules (4.16) and (4.19) produces sequences* (x_k) *and* (y_k) *in* ω *and a bounded sequence of matrices* (H_k) *with bounded inverses. Then, one of the following situations occurs:*

(i) $\liminf\,(\|c(x_k)\|+\|g(y_k)\|) = 0$,

(ii) $\mu_k = \mu$ *for k large,* (p_k) *is unbounded and* $\|\lambda(x_k)\|_D \longrightarrow \infty$ *when* $k \longrightarrow \infty$ *in* $\{\, k : p_k > p_{k-1} \,\}$,

(iii) $\mu_k = \mu$ *for k large,* $p_k = p$ *for k large and either* $l_p(x_k,\mu) \longrightarrow -\infty$ *or for some subsequence* $dist(x_k,\omega^c) \longrightarrow 0$.

Remarks. Because sequences (x_k) and (y_k) are generated by the algorithm, it is implicitly supposed that the longitudinal search algorithm succeeds at each iteration k, which will usually occur (see Theorem 3.3). Statement (i) is equivalent to *lim* $\epsilon_k^0 = 0$, so *lim inf* has not the same sense as in topology. The fact that only *lim inf* is obtained in (i) (instead of *lim*) does not come from the hypotheses of the theorem that are rather strong (the boundedness of (H_k) and of (H_k^{-1}) are usually enough to imply the convergence of all the sequence ϵ_k to zero, as this may be observed in unconstrained optimization, see Wolfe (1969, Theorem 1)), but from the way the convergence is checked by the use of ϵ_k^0 in statement 8 and 9 of the algorithm. This may be difficult to improve because, it is impossible to design an algorithm that builds a *decreasing* sequence $(\tilde{\epsilon}_k)$ from the sequence (ϵ_k) such that $(\tilde{\epsilon}_k)$ depends only on $\{\, \epsilon_i : 0 \le i \le k \,\}$ and such that $\tilde{\epsilon}_k \longrightarrow 0$ if and only if $\epsilon_k \longrightarrow 0$ (the reason of this is that ϵ_k may be built by observing $\tilde{\epsilon}_k$ like in $\{\, i := 1 \;;\; \epsilon_0 := 0 \;;$ for $k \ge 0$ do $\{$ compute $\tilde{\epsilon}_k$; if $\tilde{\epsilon}_k \ge 1/i$ then $\epsilon_{k+1} := 0$ else $\{\, \epsilon_{k+1} := 1 \;;\; i := i+1 \,\} \,\} \,\}$). The situation (ii) of the theorem means that either $\{\, x_k : p_k > p_{k-1}\}$ is unbounded or has accumulation points $\bar{x}$ on the boundary of ω at which $\lambda(\bar{x})$ is not well defined by (1.15), for instance, because $A(\bar{x})$ has not full rank. In (iii), dist(x_k,ω^c) is the distance from x_k to the complementary set of ω.

Proof of Theorem 4.3. Let us suppose that situation (i) does not occur. Then, $\liminf \epsilon_k := \epsilon_\infty^0 > 0$ and, by (4.18) and (4.21), $\underline{p}_k = \underline{p}$ for k large, $\mu_k = \mu$ for k large and (p_k) is an increasing sequence for k large. Then, either (p_k) is unbounded or (p_k) is bounded! In the first case, this means by (4.21) and (4.8) that $(\|\lambda(x_k)\|_D)$ is unbounded, and more precisely,

$$\|\lambda(x_k)\|_D \longrightarrow +\infty \quad \text{for} \quad k \longrightarrow \infty \quad \text{in} \quad \{\, k : p_k > p_{k-1}\} ,$$

which is the conclusion (ii) of the theorem. On the other hand, if (p_k) is bounded, (4.21) and (4.9) show that p_k changes finitely often. So, $p_k = p$ for k large. We prove the rest of (iii) by contradiction, supposing that $l_p(x_k,\mu)$ is bounded from below and that (x_k) remains away from ω^c. We have

$$l_p(x_{k+1},\mu) \le l_p(y_k,\mu) \le l_p(x_k,\mu) .$$

Therefore, the sequences $(l_p(x_k,\mu))$ and $(l_p(y_k,\mu))$ decrease to the same value.

According to the longitudinal displacement, i.e. to inequalities (3.15) and (3.16), we have

$$l_p(x_{k+1},\mu) - l_p(y_k,\mu) \leq \alpha_1 \tau_k \nabla f(y_k)^T t_k \leq - \alpha_1 h \tau_k ||g(y_k)||_2^2 , \tag{4.22}$$

$$\begin{aligned}(g(x_{k+1}) - g(y_k))^T Z(y_k) t_k &\geq - (1-\alpha_2) \nabla f(y_k)^T t_k \\ &\geq (1-\alpha_2) h ||g(y_k)||_2^2 ,\end{aligned} \tag{4.23}$$

where we have used the fact that, (H_k^{-1}) being bounded, there exists a positive constant h such that $H_k \geq h I$ in $\mathbb{R}^{(n-m)\times(n-m)}$. From (4.22) and the convergence of $(l_p(y_k,\mu))$ and $(l_p(x_k,\mu))$ to the same value, we deduce that $\tau_k||g(y_k)||^2 \longrightarrow 0$. From (4.23), the boundedness of (H_k), the Lipschitz continuity of g and proposition 3.4, we get

$$||g(y_k)||^2 \leq C ||x_{k+1}-y_k|| ||g(y_k)|| \leq C \tau_k ||g(y_k)||^2 .$$

Hence, $g(y_k) \longrightarrow 0$.

Now, from the transversal search (see (4.6)), we have

$$\begin{aligned}&l_p(y_k,\mu) - l_p(x_k,\mu) \\ &\quad\leq \alpha \rho_k [(\lambda(x_k) - \mu)^T c(x_k) - p ||c(x_k)||] \\ &\quad\leq - \alpha \underline{p} \rho_k ||c(x_k)|| .\end{aligned}$$

Hence, $\rho_k ||c(x_k)|| \longrightarrow 0$. We are going to show that (ρ_k) is bounded away from zero, which will prove (iii) since then $c(x_k) \longrightarrow 0$ and, with $g(y_k) \longrightarrow 0$, this gives the expected contradiction with the initial assumption, according to which $\epsilon_k^0 \geq \epsilon_\infty^0 > 0$. Again, we set about it by contradiction. So, let us suppose $\rho_k < 1$ and $\rho_k \longrightarrow 0$ for k in a subsequence $\mathbb{K}$. We may consider that ρ_k/β is not accepted by the line search because Armijo's criterion (4.6) is not satisfied. Indeed, otherwise it would mean that for a subsequence $\mathbb{K}'$ of $\mathbb{K}$, $\tilde{x}_k := x_k + (\rho_k/\beta) r_k$ would not be in ω. But $\rho_k ||c(x_k)|| \longrightarrow 0$ implies that $(\rho_k/\beta) r_k \longrightarrow 0$ and therefore $(x_k)_{k\in \mathbb{K}'}$ would go closer and closer to a point $\tilde{x}_k$ not in ω, a situation that has been discarded by hypothesis. So, for k in $\mathbb{K}$, we have

$$\begin{aligned}&l_p(x_k+\frac{\rho_k}{\beta}r_k,\mu) - l_p(x_k,\mu) \\ &\quad> \alpha \frac{\rho_k}{\beta} [(\lambda(x_k)-\mu)^T c(x_k) - p ||c(x_k)||] .\end{aligned} \tag{4.24}$$

Expanding $l_p(x_k+\frac{\rho_k}{\beta}r_k,\mu)$ about x_k gives

$$
\begin{aligned}
& l_p(x_k + \frac{\rho_k}{\beta} r_k, \mu) \\
& \quad \le f(x_k) + \frac{\rho_k}{\beta}\, \lambda(x_k)^T\, c(x_k) + \rho_k^2\, C\, ||r_k||^2 \\
& \quad + \mu^T\, c(x_k) - \frac{\rho_k}{\beta}\, \mu^T\, c(x_k) + p(1-\frac{\rho_k}{\beta})\, ||c(x_k)|| \\
& \quad \le l_p(x_k,\mu) + \frac{\rho_k}{\beta}\, [\, (\lambda(x_k)-\mu)^T\, c(x_k) - p||c(x_k)||\,] + \rho_k^2\, C\, ||r_k||^2
\end{aligned}
$$

where C is a positive constant that does not depend on k. Therefore, with (4.24) and (4.3), this leads to

$$
(1-\alpha)\, \frac{\rho_k}{\beta}\, \underline{p}\, ||c(x_k)|| < C\, \rho_k^2\, ||c(x_k)||^{\,2}\, .
$$

Therefore $\rho_k\, ||c(x_k)||$ is positive for k in $\mathbb{K}$ and dividing by this factor in this inequality, we obtain a contradiction with the fact that $\rho_k\, ||c(x_k)|| \longrightarrow 0$. □

The next theorem shows that our way to adapt $\underline{p}_k$, p_k and μ_k will guarantee the acceptance of the unit transversal and longitudinal step-sizes.

Theorem 4.4. *Suppose that algorithm RSA (4.7) with the adaptation rules (4.16) and (4.19) produces a sequence* (x_k) *in* ω *converging in* ω *to a solution* x_* *of problem (1.1) and a bounded sequence of positive definite matrices* (H_k) *with bounded inverses. Suppose, as well, that condition (4.12) is satisfied. Then, we have* $\rho_k = 1$, $l_k = 0$ *and* $\tau_k = 1$ *for* k *large enough.*

Proof. As $x_k \longrightarrow x_*$, we have $c(x_k) \longrightarrow 0$, hence $y_k \longrightarrow x_*$, $g(y_k) \longrightarrow 0$, $\epsilon_k \longrightarrow 0$ and, by (4.20), $\mu_k \longrightarrow \lambda_*$.

We begin with the longitudinal displacement. Suppose that $l_k \ne 0$ or $\tau_k \ne 1$ infinitely often. Then, by (4.17), we would have $\underline{p}_k \longrightarrow 0$ and by (4.20), (4.21) and (4.10), $p_k \longrightarrow 0$. Therefore, using proposition 4.2 (and the comments after its proof), we see that the left hand side of (4.13) and (4.14) becomes negative for k large, then $l_k = 0$ and $\tau_k = 1$ for k large: a contradiction.

Hence, $l_k = 0$ and $\tau_k = 1$ for k large and from (4.18) we see that $\underline{p}_k = \underline{p} > 0$ for k large. By using proposition 4.1 (and the comments after the proof of proposition 4.2), we see that the left hand side of (4.11) becomes negative for k large, hence $\rho_k = 1$. □

5. Conclusion

We have presented an algorithm for equality constrained optimization of the reduced type in which the projected Hessian of the Lagrangian is approximated by updating *at each iteration* a matrix G_k using the BFGS formula and two vectors γ_k, the change in the reduced gradient, and δ_k, the corresponding reduced displacement.

The main purpose of the paper is to show the possibility of obtaining the positivity of $\gamma_k^T \delta_k$, which is essential to guarantee the positive definiteness of the reduced matrices G_k. This feature is due to a particular design of the longitudinal displacement which minimizes the objective function f while roughly maintains constant the value of the constraint function c. For this, we introduce a step-size τ_k scaling the reduced displacement in $\mathbb{R}^{n-m}$ while the longitudinal displacement in $\mathbb{R}^n$ becomes piecewise linear. Wolfe's criteria are used to determine the step-size τ_k.

The algorithm is made globally convergent by using a nondifferentiable augmented Lagrangian function. Another feature of the algorithm is to separate completely the longitudinal and transversal displacements. Indeed, the step-size of both displacements are determined by two different searches on the penalty function.

The technique used here to maintain the positive definiteness of the matrices G_k may be seen as a generalization to equality constrained optimization of Wolfe's step-size selection procedure in unconstrained optimization. It is well known that this technique cannot be used in the framework of quasi-Newton or SQP methods. As the technique works well in unconstrained optimization, this may be seen as an advantage of the reduced framework over the SQP methods. However, the algorithm proposed here always requires at least two (and exactly two, asymptotically) linearizations of the constraints for each superlinear step, which can be an important overcost in some applications. Therefore, the developed technique should be extended to those reduced methods that require only one linearization of the constraints per iteration.

As mentioned in the text, a weak point of the algorithm lies in the way the multipliers and the penalty parameters are adapted to improve the penalty function. Indeed, it requires from the algorithm to feel the closeness of a solution and therefore impoverishes the global convergence result (see the remarks after Theorem 4.3). We think that some progress might be obtained on this topic as well.

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