Working Paper

Partially Observable Control Problems with Compulsory Shifts of the State

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WP-92-34
May 1992

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FOREWORD

Stochastic control, and more generally stochastic optimization, deals with problems where decisions have to be taken in the face of uncertainty and these decisions have to be optimal or at least nearly optimal — in some specific sense. "Stochastic" means that uncertainty is described in a probabilistic setting. More specifically, stochastic control deals with dynamic optimization problems, namely problems where the decisions (also called controls) affect the (stochastic) evolution over time of a given system. In many applications, the time horizon, over which we control a given system, is very remote; for the mathematical description it is therefore infinite. As optimality criterion according to which we choose the decisions (controls) one then takes the minimization of the long-run average cost.

The present paper concerns such infinite horizon stochastic control problems with the average cost criterion, when the evolution of the controlled system is furthermore observed only on an incomplete basis. Conditions are given under which it is possible to actually compute nearly optimal decisions (controls) for such problems.

ABSTRACT

Stochastic control problems with partial state observation and the long-run average cost criterion are among the most difficult dynamic stochastic optimization problems and almost nothing has so far appeared in the literature concerning their solution. On the other hand many problems in Engineering, Operations Research, and the Economic and Social Sciences can be modelled as problems of the above type. In the present paper we study conditions under which the filtering process associated with the partially observed state process has a unique invariant measure and describe ways to approximate it. We finally discuss the applications of these results to the construction of nearly optimal controls.

PARTIALLY OBSERVABLE CONTROL PROBLEMS WITH COMPULSORY SHIFTS OF THE STATE

Wolfgang J. Runggaldier and Lukasz Stettner

INTRODUCTION

A common approach for the study of stochastic control problems under partial observation of the state is to consider the so-called separated problem. This problem is of the form of a stochastic control problem with complete state observation and is obtained by replacing the original state process by the filter process, namely the process whose values are the conditional (normalized or unnormalized) distributions of the original state, given past and current observations. When studying stochastic control problems with partial state observation over an infinite horizon and with the long-run average cost criterion, the ergodic properties of the filtering process become most important. In particular, for approximation purposes it is crutial to have a unique invariant measure. The properties of the filtering process depend in turn on properties of the original state process as well as the observation process. The main purpose of this study is to give conditions on the original control model to ensure the existence of a unique invariant measure. Results on convergence of invariant measures will also be given together with possible applications to the construction of nearly optimal controls. A related study was already performed by the authors in [3] where the conditions imposed on the the original control model, to ensure uniqueness of the invariant measure for the filtering process, implied both a restriction of the class of admissible controls as well as conditions on the transition kernel of the original state process. Here we take a global control approach considering, besides continuously acting controls, also compulsory periodic shifts of the state on the basis of the values of the filtering process. This setting allows to improve the ergodic behaviour of the controlled filtering process and makes it possible to relax some of the assumptions in [3].

1. PROBLEM FORMULATION AND PRELIMINARY RESULTS

1a. Problem formulation

On some probability space (Ω, F, P) consider a controlled discrete time Markov process (x_i^u), $i = 1, 2, \ldots$, with values in a locally compact (but noncompact) separable state space E. Assume (x_i^u) starts from an initial law μ and denote by $P^{u_i}(x, dz)$ its transition kernel in the generic period i, where u_i represents the control that takes values in a compact set of control parameters $U \subset R$. The process (x_i^u) is only partially observed through an observation process (y_i), $y_i \in R^d$, defined by

$$y_i = h(x_i) + w_i \tag{1.1}$$

where $h \in C(E, R^d)$, the space of continuous and bounded functions from E into R^d, and w_i are i.i.d. d-dimensional standard Gaussian random variables independent of x_k kor $k \leq i$. We assume that each u_i is adapted to the observation σ-algebra $Y^i = \sigma\{y_1, \ldots, y_i\}$. Given a bounded Borel function φ on E, define the filtering process, corresponding to the controlled state process (x_i^u) with observation (1.1) as follows: It is the process π_i^u with values in the space $P(E)$ of probability measures on E, endowed with the weak convergence topology, defined through

$$\begin{aligned} \pi_0^u(\varphi) &= \mu(\varphi) \\ \pi_i^u(\varphi) &= E_\mu\{\varphi(x_i^u) \mid Y^i\} \qquad i = 1, 2, \ldots \end{aligned} \tag{1.2}$$

E_μ stands for the expectation, given the initial law μ for the process (x_i^u), where the latter is controlled by a law u to be defined later; $\pi(\varphi)$ denotes the integral of a function $\varphi(x)$ with respect to the measure $\pi(dx)$.

We now define a global control strategy ϑ as given by the quadruplet $\vartheta = (u, K, \gamma, \eta)$, where

$u \in B(P(E), U)$, the set of Borel measurable function from $P(E)$ into the compact set U, and will be referred to us the "continuous control",

$K \subset E$ is compact with Int $(\partial K) = \emptyset$, Int $(K) \neq \emptyset$ and may be called "test set",

$\gamma \in (0, 1)$ is a threshold level,

$\eta \in H$ a fixed compact subset of $P(E)$, and represents the "shift measure". (In particular applications, H may be the set of Dirac δ-measures of points of a compact subset of E or the set of all probability measures over a compact subset of E).

REMARK. Instead of the strategy ϑ as defined above, we may also consider the quadruplets of the form $\vartheta = (u, \psi, \gamma, \eta)$, where u, γ, η are as before, while $\psi \in C(E)$, has compact support, and satisfies $0 \leq \psi \leq 1$.

The results obtained below for strategies $\vartheta = (u, K, \gamma, \eta)$ can easily be carried over to the case considered here, by just replacing, for any measure $\nu \in P(E)$, $\nu(K)$ by $\nu(\psi) = \int \psi(x)\nu(dx)$.

Having defined the strategy ϑ, from now on, instead of (x_i^u) and π_i^u, we shall more appropriately use the notation (x_i^ϑ) and (π_i^ϑ) respectively to denote the state and filter processes governed by the more general strategy ϑ. The effect of ϑ on the evolution of the state and filter processes is as follows:

The continuous-control component u of ϑ, which through the kernel $P^u(x, dz)$ governs the evolution of the state process (x_i^ϑ), is defined in the generic period i by the relations:

$$u_i = \begin{cases} u(\pi_i^\vartheta) & \text{if } \pi_i^\vartheta(K) \geq \gamma \\ u(\eta) = \bar{u} & \text{if } \pi_i^\vartheta(K) < \gamma \end{cases} \tag{1.3}$$

where $u \in A = C(P(E), U)$. In words, if at stage i we have $\pi_i^\vartheta(K) \geq \gamma$, then the control u_i is Y^i-adapted in the sense specified by the first alternative in (1.3), otherwise, a compulsory shift is applied to the state x_i^ϑ, restarting its evolution from a given fixed measure $\eta \in H \subset P(E)$, independently of past history, and applying a control value $\bar{u} = u(\eta)$.

Coming to the filter process, consider, for fixed ϑ, the sequence of measures

$$\hat{\pi}_{i+1}^\vartheta(dz) = \begin{cases} \int_E P^{u(\pi_i^\vartheta)}(x, dz)\pi_i^\vartheta(dx) & \text{if } \pi_i^\vartheta(K) \geq \gamma \\ \int_E P^{\bar{u}}(x, dz)\eta(dx) & \text{if } \pi_i^\vartheta(K) < \gamma \end{cases} \tag{1.4}$$

which represents the distribution, under ϑ of x_{i+1}^ϑ given π_i^ϑ.

Using then the so-called measure transformation approach to filtering, we obtain that the controlled filtering process (π_i^ϑ) satisfies, for given ϑ and $\varphi \in B(E)$

$$\begin{aligned} &\pi_{i+1}^\vartheta(\varphi) = R(y_{i+1}, \hat{\pi}_i^\vartheta)(\varphi) := \\ &:= \frac{\int_E \varphi(z)\exp[(y_{i+1}, h(z)) - \frac{1}{2}(h(z), h(z))]\hat{\pi}_i^\vartheta(dz)}{\int_E \exp[(y_{i+1}, h(z)) - \frac{1}{2}(h(z), h(z))]\hat{\pi}_i^\vartheta(dz)} \end{aligned} \tag{1.5}$$

where we implicitly define the operator R. It is easily seen that, under a given ϑ, the controlled filtering process (π_i^ϑ) is Markov. Let $\Pi^\vartheta(\mu, \Lambda)$ denote its transition kernel, where $\Lambda \in B(P(E))$, the σ-field of Borel subsets of $P(E)$.

We shall be interested in control over an infinite horizon, minimizing a long run average cost criterion; given bounded Borel functions $c : E \times U \to R$ and $d : E \times P(E) \to R^+$ we then consider as objective function the cost functional

$$J_\mu(\vartheta) = \limsup_{n\to\infty} n^{-1} E_\mu \left\{ \sum_{i=0}^{n-1} c(x_i^\vartheta, u(\pi_i^\vartheta)) \right.$$
$$\left. + \sum_{i=0}^{n-1} d(x_i^\vartheta, \eta)\chi_{\pi_i^\vartheta(K)<\gamma} \right\} = \tag{1.6}$$
$$= \limsup_{n\to\infty} n^{-1} E_\mu \left\{ \sum_{i=0}^{n-1} C(\pi_i^\vartheta) + \sum_{i=0}^{n-1} D(\pi_i^\vartheta, \eta)\chi_{\pi_i(K)<\gamma} \right\}$$

having put

$$C(\nu) = \int_E c(z, u(\nu))\nu(dz) \qquad D(\nu, \eta) = \int_E d(z, \eta)\nu(dz) \tag{1.7}$$

1b. Assumptions

We shall make the following assumptions

(A1) There exists $j \in \{1, 2, \ldots, d\}$ such that the j-th component $h^j(x)$ of $h(x)$ has a limit at "∞" and attains at "∞" either its strong maximum or strong minimum. More precisely, letting

$$K_n = \{x \in E : \ \rho(x, \bar{x}) \leq n\}$$

where ρ is a metric on E compatible with the topology and $\bar{x}$ is a fixed element of E, we either have

$$\sup_{x\in K_n} h^j(x) < \sup_{x\in E} h^j(x) \qquad \text{for } n = 1, 2, \ldots$$

or

$$\inf_{x\in K_n} h^j(x) > \inf_{x\in E} h^j(x)$$

(A2) For any compact set $K \subset E$ there exists $\alpha > 0$ such that

$$\inf_{v\in U} \inf_{x\in E} P^v(x, K^c) \geq \alpha.$$

Notice that (A2) represents a nondegeneracy condition for the state process that is always satisfied for state evolution models with nondegenerate additive Gaussian noise.

Additional assumptions will be formulated in Section 3 where convergence of invariant measures is considered.

1c. Preliminary results and background

Given a control strategy ϑ and initial measure $\mu \in P(E)$, let

$$\tau := \inf\{i > 0 \mid (\pi_i^\vartheta)(K) < \gamma\}. \tag{1.8}$$

Lemma 1.1. *Under (A1), (A2) we have*

$$\inf_\vartheta \inf_{\mu \in P(E)} P_\mu\{(\pi_i^\vartheta)(K) < \gamma\} = \beta > 0. \tag{1.9}$$

PROOF. Can be obtained along the lines of Lemma 2.1 and Corollary 2.2 in [3].

■

Corresponding to Corollary 2.3 in [3] we then have

Corollary 1.1. *Under (A1), (A2) we have for all $k = 1, 2, \ldots$ and all strategies ϑ*

$$\sup_{\mu \in P(E)} E_\mu(\tau^k) < \infty. \tag{1.10}$$

Consider now the sequence of stopping times

$$\tau_1 = \tau;\ \tau_2 = \tau_1 + \tau \circ \Theta_{\tau_1}, \ldots, \tau_{n+1} = \tau_n + \tau \circ \Theta_{\tau_n} \tag{1.11}$$

where $\eta \in P(E)$ is the fixed restart (shift) measure introduced in Section 1.1 and Θ_{τ_n} stands for the Markov shift operator of the filtering process.

Notice that $\{x^\vartheta_{\tau_i+1},\ i = 1, 2, \ldots\}$ form a sequence of i.i.d. random variables with common law (cf (1.4))

$$\hat{\pi}^\vartheta(dz) = \int_E P^{\bar{u}}(x, dz)\eta(dx) \tag{1.12}$$

Defining, furthermore

$$y_{\tau_i+1} = h(x_{\tau_i+1}) + w_{\tau_i+1} \tag{1.13}$$

where w_{τ_i+1}, $(i = 1, 2, \ldots)$ are i.i.d. standard Gaussian we have (cf (1.5))

$$\pi^\vartheta_{\tau_i+1}(\varphi) = R(y_{\tau_i+1}, \hat{\pi}^\vartheta)(\varphi) \tag{1.14}$$

from which it is easily seen that, analogously to $\{x^\vartheta_{\tau_i+1},\ (i = 1, 2, \ldots)\}$, $\{\pi^\vartheta_{\tau_i+1},\ i = 1, 2, \ldots\}$ is an i.i.d. sequence of random variables with values in $P(E)$.

As will become apparent from the next section, the fact of having a sequence of stopping times τ_i, $i = 1, 2, \ldots$ with finite moments, leading to the i.i.d. sequence $\{\pi^{\vartheta}_{\tau_{i+1}},\ i = 1, 2, \ldots\}$ will allow us to obtain a unique invariant measure by means of the Strong Law of Large Numbers for martingales. The above fact is here obtained from the introduction of the global control strategy ϑ, combined with assumption (A2): The strategy ϑ considers compulsory shifts by periodically restarting, on the basis of the current value of the filter, the evolution of the state process from the fixed measure η, while assumption (A2) guarantees that shifts are applied at successive intervals, whose expected duration is bounded uniformly with respect to the initial measure.

Uniqueness of the invariant measure for the filtering process was obtained also in [3]. There, the set of admissible strategies of the form $u_i = u(\pi^u_i)$ with $u \in C(P(E), U)$ was restricted to a subclass that considers a periodic return, at stopping times τ_i to the fixed control value $u = 0$, based on the current value of the filter. An assumption corresponding to (A2) then guarantees that return to $u = 0$ occurs at successive intervals, whose duration is uniformly bounded. To obtain then an i.i.d. sequence $\{\pi_{\tau_{i+1}},\ i = 1, 2, \ldots\}$, in [3] we had to introduce a restriction on the transition kernel of the original state process, by requiring that, for a control value $u = 0$, we have $P^u(x, dz) = \eta(dz)$ for all $x \in E$ with $\eta \in P(E)$ given.

2. ERGODIC BEHAVIOUR OF THE CONTROLLED FILTER PROCESS (EXISTENCE OF A UNIQUE INVARIANT MEASURE)

The main result of this section is

Theorem 2.1. *Under (A1), (A2) and given a strategy ϑ, there exists a unique invariant measure ϕ^{ϑ} for the controlled filtering process (π^{ϑ}_i). The measure ϕ^{ϑ} has the following representation, where $F \in B(P(E))$ the set of Borel bounded functions on $P(E)$ and η is the given shift measure*

$$\phi^{\vartheta}(F) = E_{\eta}\left\{\sum_{i=1}^{\tau} F(\pi^{\vartheta}_i)\right\}\{E_{\eta}(\tau)\}^{-1} \tag{2.1}$$

Moreover, for all $\mu \in P(E)$

$$\begin{aligned} J_{\mu}(\vartheta) &= \int_{P(E)} C(\nu)\phi^{\vartheta}(d\nu) + \int_{P(E)} D(\nu,\eta)\chi_{\nu(K)<\gamma}\phi^{\vartheta}(d\nu) \\ &= \frac{E_{\eta}\left\{\sum_{i=1}^{\tau} C(\pi^{\vartheta}_i) + D(\pi^{\vartheta}_{\tau},\eta)\right\}}{E_{\eta}(\tau)} \end{aligned} \tag{2.2}$$

where $C(\nu)$ and $D(\nu,\eta)$ are as in (1.7).

In the rest of this section we derive some auxiliary results that in the end will allow us to obtain rather immediately the proof of Theorem 2.1.

Given a bounded and measurable function $K(\cdot)$, $K : P(E) \to R$, define for a given strategy ϑ a process $\{z_n\}_{n=1,2,\ldots}$ as

$$z_n := \sum_{i=1}^{\tau_n} K(\pi_i^\vartheta) - nE_\eta\left(\sum_{i=1}^{\tau} K(\pi_i^\vartheta)\right) \tag{2.3}$$

where τ and τ_n are as defined in (1.8) and (1.11) respectively, η is the given shift measure introduced in Section 1.1, and (π_i^ϑ) denotes the filtering process starting from an initial measure $\mu \in P(E)$ and evolving under the strategy ϑ.

Lemma 2.1. *Under (A1), (A2) and taking as initial measure for (π_i^ϑ) the measure $\mu = \eta$ we have that the process $\{z_n\}$ is a square integrable $G_n = \sigma\{y_1, \ldots, y_{\tau_n}\}$ martingale with*

$$\sum_{n=1}^{\infty} \frac{E(z_n - z_{n-1})^2}{n^2} < \infty \tag{2.4}$$

PROOF. Notice that

$$z_n - z_{n-1} = \sum_{i=\tau_{n-1}+1}^{\tau_n} K(\pi_i^\vartheta) - E_\eta\left\{\sum_{i=1}^{\tau} K(\pi_i^\vartheta)\right\} \tag{2.5}$$

Thus from the definition of τ_n (see 1.11) we have

$$E[z_n - z_{n-1} | G_{n-1}] = 0 \tag{2.6}$$

and from (1.10)

$$E\{|z_n - z_{n-1}|^2 \mid G_{n-1}\} \leq C < \infty \tag{2.7}$$

which is sufficient for (2.4) to hold.

∎

Given the result of Lemma 2.1, we can apply the Law of Large Numbers for martingales (see Thm VII.8.2 of [2]) obtaining

Corollary 2.1. *Under the assumptions of Lemma 2.1 but letting the initial measure for π_i^ϑ be an arbitrary $\mu \in P(E)$ we have*

$$\lim_{n\to\infty} \frac{z_n}{n} = 0 \qquad P \text{ a.s} \tag{2.8}$$

The following two corollaries are obtained from Corollary 2.1 by particularizing the function $K(\cdot)$ in (2.3).

Taking $K(\cdot) \equiv 1$ we have

Corollary 2.2. *The assumptions are those of Corollary 2.1. Then*

$$\lim_{n\to\infty} \frac{\tau_n}{n} = E_\eta(\tau) \qquad P \text{ a.e.} \tag{2.9}$$

Taking $K(\nu) = C(\nu) + D(\nu,\eta)\chi_{\nu(K)<\gamma}$ with C and D as in (1.7) we have

Corollary 2.3. *Again with the assumptions of Corollary 2.1*

$$\begin{aligned}&\lim_{n\to\infty} n^{-1}\left\{\sum_{i=1}^{\tau_n}\left[C(\pi_i^\vartheta) + D(\pi_i^\vartheta,\eta)\chi_{\pi_i^\vartheta(K)<\gamma}\right]\right\}\\ &= E_\eta\left\{\sum_{i=1}^{\tau} C(\pi_i^\vartheta) + D(\pi_\tau^\vartheta,\eta)\right\} \quad P \text{ a.s.}\end{aligned} \tag{2.10}$$

Lemma 2.2. *For bounded Borel $K : P(E) \to R$ and under (A1), (A2) we have for all ϑ*

$$\liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n} K(\pi_i^\vartheta) = \liminf_{n\to\infty} \tau_n^{-1}\sum_{i=1}^{\tau_n} K(\pi_i^\vartheta) \tag{2.11}$$

with the same relation holding also for $\limsup$ *instead of* $\liminf$. *The initial measure for the filtering process π_i^ϑ may be any measure $\mu \in P(E)$.*

PROOF. Defining

$$\beta(n) := \max\{i \mid \tau_i < n\} \tag{2.12}$$

we have

$$\begin{aligned}&n^{-1}\left(\sum_{i=1}^{n} K(\pi_i^\vartheta)\right) = n^{-1}\left[\sum_{i=1}^{\tau_{\beta(n)}} K(\pi_i^\vartheta) + \sum_{i=\tau_{\beta(n)}+1}^{n} K(\pi_i^\vartheta)\right]\\ &= \frac{\tau_{\beta(n)}}{n}\cdot\frac{1}{\tau_{\beta(n)}}\left(\sum_{i=1}^{\tau_{\beta(n)}} K(\pi_i^\vartheta)\right) + \frac{1}{n}\left(\sum_{i=\tau_{\beta(n)}+1}^{n} K(\pi_i^\vartheta)\right) \leq\\ &\leq \frac{\tau_{\beta(n)}}{n}\cdot\frac{1}{\tau_{\beta(n)}}\left(\sum_{i=1}^{\tau_{\beta(n)}} K(\pi_i^\vartheta)\right) + \frac{\|K\|(\tau_{\beta(n)+1} - \tau_{\beta(n)})}{n}\end{aligned} \tag{2.13}$$

Now, noticing that $\{\beta(n)\}$ is a subsequence of $\{n\}$

$$\begin{aligned}&\limsup_{n\to\infty}\frac{\tau_{\beta(n)+1} - \tau_{\beta(n)}}{n} \leq \limsup_{n\to\infty}\frac{\tau_{\beta(n)+1} - \tau_{\beta(n)}}{\beta(n)}\cdot\frac{\beta(n)}{n}\\ &\leq \limsup_{n\to\infty}\frac{\tau_{n+1} - \tau_n}{n} = \limsup_{n\to\infty}\left[\frac{\tau_{n+1} - (n+1)E_\eta(\tau)}{n+1}\frac{n+1}{n}\right.\\ &\left. - \frac{\tau_n - nE_\eta(\tau)}{n} + \frac{E_\eta(\tau)}{n}\right] = 0 \qquad \text{P a.s.}\end{aligned} \tag{2.14}$$

the last equality being a consequence of Corollary 2.2 and the finitness of $E_\eta(\tau)$ (Corollary 1.1).

On the other hand, since

$$\left|\frac{\tau_{\beta(n)}}{n} - 1\right| = \frac{n - \tau_{\beta(n)}}{n} \leq \frac{\tau_{\beta(n)+1} - \tau_{\beta(n)}}{n} \tag{2.15}$$

from (2.14) we also have

$$\lim_{n\to\infty} \frac{\tau_{\beta(n)}}{n} = 1 \qquad \text{P a.s.} \tag{2.16}$$

From (2.13), (2.14), (2.16), noticing that $\{\tau_n\}$ is a subsequence of $\{n\}$, and $\{\tau_{\beta(n)}\}$ is in turn a subsequence of $\{\tau_n\}$, we have

$$\begin{aligned} \liminf_{n\to\infty} n^{-1}\left(\sum_{i=1}^{n} K(\pi_i^\vartheta)\right) &\leq \liminf_{n\to\infty} \tau_n^{-1}\left(\sum_{i=1}^{\tau_n} K(\pi_i^\vartheta)\right) \\ \leq \liminf_{n\to\infty} \tau_{\beta(n)}^{-1} \sum_{i=1}^{\tau_{\beta(n)}} K(\pi_i^\vartheta) &= \liminf_{n\to\infty} n^{-1}\left(\sum_{i=1}^{n} K(\pi_i^\vartheta)\right) \end{aligned} \tag{2.17}$$

i.e. we have obtained (2.11). Analogous procedure holds for $\limsup\limits_{n\to\infty}$.

■

Corollary 2.4. *Under the assumptions of Lemma 2.2 we have*

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} K(\pi_i^\vartheta) = \frac{1}{E_\eta(\tau)} E_\eta\left\{\sum_{i=1}^{\tau} K(\pi_i^\vartheta)\right\} \tag{2.18}$$

PROOF. From Lemma 2.2, using also Corollary 2.2 and Corollary 2.1 we have

$$\begin{aligned} \liminf_{n\to\infty} n^{-1}\left(\sum_{i=1}^{n} K(\pi_i^\vartheta)\right) &= \liminf_{n\to\infty} \tau_n^{-1}\left(\sum_{i=1}^{\tau_n} K(\pi_i^\vartheta)\right) = \\ = \liminf_{n\to\infty} \frac{n}{\tau_n}\frac{1}{n} \sum_{i=1}^{\tau_n} K(\pi_i^\vartheta) &= \frac{1}{E_\eta(\tau)} E_\eta\left\{\sum_{i=1}^{\tau} K(\pi_i^\vartheta)\right\} \end{aligned} \tag{2.19}$$

Since also (Lemma 2.2)

$$\limsup_{n\to\infty} n^{-1} \sum_{i=1}^{n} K(\pi_i^\vartheta) = \frac{1}{E_\eta(\tau)} E_\eta\left\{\sum_{i=1}^{\tau} K(\pi_i^\vartheta)\right\} \tag{2.20}$$

we obtain (2.18).

PROOF OF THEOREM 2.1. Notice first that (2.18) can be rewritten as

$$\lim_{n\to\infty} n^{-1} E_\mu \left\{ \sum_{i=1}^{n} K(\pi_i^\vartheta) \right\} = \frac{E_\eta\left(\sum_{i=1}^{\tau} K(\pi_i^\vartheta)\right)}{E_\eta(\tau)} := \Phi(K) \tag{2.21}$$

valid for any $\mu \in P(E)$ and where we implicitly define a measure $\Phi \in P(P(E))$. From (2.21) we first obtain the invariance of the measure Φ ; in fact (2.21) also holds when replacing K by $\Pi^\vartheta K$ where (see Section 1.1) Π^ϑ denotes the transition kernel of the filter Markov process (π_i^ϑ) (notice that K was required only to be bounded and measurable). From (2.21) however we also obtain the uniqueness of the invariant measure Φ. Assume in fact that Ψ is another invariant measure; then for all n we have

$$\int n^{-1} \sum_{i=1}^{n} E_\mu(K(\pi_i^\vartheta))\Psi(d\mu) = \int K(\mu)\Psi(d\mu) \tag{2.22}$$

On the other hand letting $n \to \infty$, from (2.21) and Dominated Convergence Theorem we have

$$\begin{aligned} &\lim_{n\to\infty} \int n^{-1} \sum_{i=1}^{n} E_\mu(K(\pi_i^\vartheta))\Psi(d\mu) = \\ &\int \Phi(K)\Psi(d\mu) = \Phi(K) = \int K(\mu)\Phi(d\mu) \end{aligned} \tag{2.23}$$

From the equality of the right hand sides in (2.22) and (2.23) and the arbitrariness of the bounded and measurable function K we obtain the uniqueness of the invariant measure.

The representation (2.2) is an immediate consequence of the preceding results by taking $K(\cdot)$ as for Corollary 2.3.

■

3. APPROXIMATIONS OF THE INVARIANT MEASURE

In this section we shall use v to denote the generic element in the set U of control parameters.

In order to approximate the invariant measure ϕ^{ϑ} of the filtering process corresponding to the original control model with state process (x_i^{ϑ}), observation function h (see (1.1)), and control strategy $\vartheta = (u, K, \gamma, \eta)$, we start by approximating (x_i^{ϑ}). For this purpose let the approximating process $(x_i^{m,\vartheta})$ $(m = 1, 2, \ldots)$ be Markov with transition kernel $P_m^{\nu}(x, dz)$ such that

$$\begin{aligned} &\text{if } U \ni v_m \to v, \text{ then } P_m^{v_m}(x, \cdot) \Rightarrow P^v(x, \cdot) \\ &\text{uniformly in } x \text{ from compact subsets of } E. \end{aligned} \tag{3.1}$$

Concerning the observations, let $h(\cdot)$ in (1.1) be approximated by functions $h_m \in B(E)$ such that

$$\sup_{x \in E} |h_m(x) - h(x)| \to 0 \text{ for } m \to \infty \tag{3.2}$$

Finally, approximate any given control strategy $\vartheta = (u, K, \gamma, \eta)$ by strategies $\vartheta_m = (u_m, K_m, \gamma_m, \eta_m)$ where

$$u_m(\nu) \to u(\nu) \text{ as } m \to \infty; \ u_m \in B(P(E), U) \tag{3.3}$$

the convergence holding uniformly in ν from compact subsets of $P(E)$;

$$\max\{\sup_{x \in K} \rho(x, K_m), \ \sup_{x \in K_m} \rho(x, K)\} \overset{m \to \infty}{\longrightarrow} 0; \tag{3.4}$$

K_m compact in E; namely, $K_m \to K$ in the Hausdorff metric (ρ is the distance on E);

$$\gamma_m \overset{m \to \infty}{\longrightarrow} \gamma; \ \eta_m \overset{m \to \infty}{\Longrightarrow} \eta \tag{3.5}$$

where $\Longrightarrow$ denotes weak convergence.

The main result of this section is Theorem 3.1 below, for which, besides (A1), (A2) we also need the following assumptions

(A3) For fixed $v \in U$, the transition kernel $P^v(x, \cdot)$ is Feller

(A4) If $U \ni v_m \to v$, then $P^{v_m}(x, \cdot) \Longrightarrow P^v(x, \cdot)$ uniformly for x from compact subsets of E i.e. for any $f \in C(E)$
$P^{v_m} f(x) \to P^v f(x)$, uniformly on compact subsets of E

(A5) If $K = \bar{K}$ and Int $(K) = \emptyset$, then $\forall x \in E, \ \forall v \in U$ we have $P^v(x, K) = 0$

Theorem 3.1. *Assume (A1) – (A5) with (A2) holding uniformly in m for the sequence $P_m^v(x,\cdot)$. There exists m_0 such that, for $m > m_0$, we have a unique invariant measure $\phi_m^{\vartheta_m}$ of the filtering process (π_i^{m,ϑ_m}) that corresponds to the transition kernel P_m, observation function h_m and strategy ϑ_m. Furthermore, if (3.1) – (3.5) hold, then*

$$\phi_m^{\vartheta_m} \Rightarrow \phi^{\vartheta} \tag{3.6}$$

PROOF. We may choose m_0 sufficiently large so that, for $m > m_0$, (A1) is satisfied also with h replaced by h_m. From Theorem 2.1 we then have, for $F \in B(P(E))$

$$\phi_m^{\vartheta_m}(F) = E_{\eta_m}\left\{\sum_{i=1}^{\tau_m} F(\pi_i^{m,\vartheta_m})\right\}(E_{\eta_m}(\tau_m))^{-1} \tag{3.7}$$

where

$$\tau_m := \inf\{i > 0 \mid \pi_i^{m,\vartheta_m}(K_m) < \gamma_m\} \tag{3.8}$$

Since (A1) holds with $m > m_0$, also for h_m and (A2) holds uniformly with respect to m, we may write for $m > m_0$,

$$\begin{aligned}&\inf_{\mu\in P(E)} P_\mu\{\pi_1^{m,\vartheta_m}(K_m) < \gamma_m\} \geq \inf_{\mu\in P(E)} P_\mu\{\pi_1^{m,\vartheta_m}(B_n) < \gamma_m\} \geq \\ &\inf_{\mu\in P(E)} P_\mu\left\{\pi_1^{m,\vartheta_m}(B_n) < \frac{\gamma}{2}\right\} > \alpha > 0\end{aligned} \tag{3.9}$$

where for fixed $\bar{x} \in E$

$B_n = \{x \in E \mid \rho(x,\bar{x}) < n\}$ and n is sufficiently large so that $\cup_m K_m \subset B_n$. Notice also that (3.9) holds for any choice of ϑ_m. In line with Lemma 1.1 and Corollary 1.1 we then have for all $k = 1,2,\dots$ and all strategies ϑ^m

$$\sup_{m>m_0}\sup_{\mu\in P(E)} E_\mu\{(\tau_m)^k\} < \infty \tag{3.10}$$

From (3.7) we not only have uniqueness, but also a representation formula for $\phi_m^{\vartheta_m}$, which is completely analogous to that for ϕ^{ϑ} in (2.1). Based on this representation and the uniform in m boundedness of the moments of τ_m (see (3.10)), to prove the second part of the theorem, namely to obtain (3.6) it will be sufficient to show, for $F \in C(P(E))$ and $n \in N$

$$\begin{aligned}&E_{\eta_m}\{\chi_{\pi_1^{m,\vartheta_m}(K_m)>\gamma_m}\cdots\chi_{\pi_n^{m,\vartheta_m}(K_m)>\gamma_m}F(\pi_{n+1}^{m,\vartheta_m})\}\\ &\to E_\eta\{\chi_{\pi_1^{\vartheta}(K)>\gamma}\cdots\chi_{\pi_n^{\vartheta}(K)>\gamma}F(\pi_{n+1}^{\vartheta})\}\end{aligned} \tag{3.11}$$

Let, for $\nu_1, \ldots, \nu_{n+1} \in P(E)$ $(n \in N)$

$$\begin{cases} g_m(\nu_1, \ldots, \nu_{n+1}) = \chi_{\nu_1(K_m) > \gamma_m} \cdots \chi_{\nu_n(K_m) > \gamma_m} F(\nu_{n+1}) \\ g(\nu_1, \ldots, \nu_{n+1}) = \chi_{\nu_1(K) > \gamma} \cdots \chi_{\nu_n(K) > \gamma} F(\nu_{n+1}) \end{cases} \tag{3.12}$$

and define

$$\begin{aligned} \Gamma := \{(\nu_1, \ldots, \nu_{n+1}) \in (P(E))^{n+1} \mid \exists(\nu_1^m, \ldots, \nu_{n+1}^m) \\ \text{with } \nu_i^m \Rightarrow \nu_i \text{ such that } g_m(\nu_1^m, \ldots, \nu_{n+1}^m) \not\to g(\nu_1, \ldots, \nu_{n+1})\} \end{aligned} \tag{3.13}$$

Provided now that

$$P_\eta\{(\pi_1^\vartheta, \ldots, \pi_{n+1}^\vartheta) \in \Gamma\} = 0 \tag{3.14}$$

by Theorem I 5.5 of [1] we have that, in order to show (3.11), it will in turn be sufficient to show

$$(\pi_1^{m,\vartheta_m}, \ldots, \pi_{n+1}^{m,\vartheta_m}) \Rightarrow (\pi_1^\vartheta, \ldots, \pi_{n+1}^\vartheta) \tag{3.15}$$

where the initial law for π_i^{m,ϑ_m} is η_m while for π_i^ϑ it is η and the convergence is in the sense of weak convergence of the sequence of measure valued random variables $(\pi_1^{m,\vartheta_m}, \ldots, \pi_{n+1}^{m,\vartheta_m})$.

To obtain (3.14), notice first that

$$\begin{aligned} \Gamma \subset \{(\nu_1, \ldots, \nu_{n+1}) : \exists(\nu_1^m, \ldots, \nu_{n+1}^m) \text{with } \nu_i^m \Rightarrow \nu_i \\ \text{and for some } i = 1, 2, \ldots, n,\ \nu_i^m(K_m) \not\to \\ \nu_i(K)\} \cup \{(\nu_1, \ldots, \nu_{n+1}), \text{ for some } i = 1, 2, \ldots, n,\ \nu_i(K) = \gamma\} = \tilde{\Gamma} \end{aligned} \tag{3.16}$$

In fact if $(\nu_1, \ldots, \nu_{n+1}) \in \tilde{\Gamma}^c$, then for $i = 1, 2, \ldots, n$, $\nu_i^m(K_m) \to \nu_i(K) \neq \gamma$ and thus

$$h_m(\nu_1^m, \ldots, \nu_{n+1}^m) \to h(\nu_1, \ldots, \nu_{n+1})$$

It follows that it suffices to prove (3.14) for Γ replaced by $\tilde{\Gamma}$. For this purpose, as well as for later use we show

Lemma 3.1. *Let $K_m \to K$ in the Hausdorff metric, namely according to (3.4). If $\nu_i^m \Rightarrow \nu_i$ and $\nu_i(\partial K) = 0$, for $i = 1, 2, \ldots, n+1$, $n \in N$, then*

$$\nu_i^m(K_m) \to \nu_i(K) \tag{3.17}$$

PROOF. Let for $\delta > 0$

$$\begin{aligned} B(K, \delta) &= \{x \in E,\ \rho(x, K) \leq \delta\} \\ I(K, \delta) &= \{x \in K,\ \rho(x, K^c) \geq \delta\} \end{aligned} \tag{3.18}$$

By the definition of the Hausdorff metric, for each $\delta > 0$ we then have

$$I(K,\delta) \subset K_m \subset B(K,\delta) \tag{3.19}$$

Therefore

$$\nu_i^m(I(K,\delta)) < \nu_i^m(K_m) \leq \nu_i^m(B(K,\delta)) \tag{3.20}$$

Since one can choose $\delta > 0$ such that

$$\nu_i(\partial I(K,\delta)) = \nu_i(\partial B(K,\delta)) = 0$$

and given the weak convergence $\nu_i^m \Rightarrow \nu_i$, letting $m \to \infty$ in (3.20) we obtain

$$\begin{aligned} &\nu_i(I(K,\delta)) \leq \liminf_{m\to\infty} \nu_i^m(K_m) \leq \limsup_{m\to\infty} \nu_i^m(K_m) \\ &\leq \nu_i(B(K,\delta)) \end{aligned} \tag{3.21}$$

Letting $\delta \downarrow 0$ over values $\delta > 0$ for which $\nu_i(\partial I(K,\delta)) = \nu_i(\partial B(K,\delta)) = 0$, since by assumption $\nu_i(\partial K) = 0$, we have

$$\lim_{\delta\downarrow 0} \nu_i(I(K,\delta)) = \lim_{\delta\downarrow 0} \nu_i(B(K,\delta)) = \nu_i(K) \tag{3.22}$$

From (3.21) and (3.22) we obtain then (3.17).

∎

From the definition of $\tilde{\Gamma}$ and Lemma 3.1 we now have that

$$\begin{aligned} &P_\eta\{(\pi_1^\vartheta, \ldots, \pi_{n+1}^\vartheta) \in \tilde{\Gamma}\} \leq \\ &\leq \sum_{i=1}^n \left(P_\eta\{\pi_i^\vartheta(\partial K) > 0\} + P_\eta\{\pi_i^\vartheta(K) = \gamma\}\right) = 0 \end{aligned} \tag{3.23}$$

where the last equality follows from (A5), the fact that Int (∂K) was assumed to be empty as well as by a siutably adapted version of Lemma 2.4 in [3].

It remains to show (3.15), which will be proved by induction on n.

Letting $F \in C(P(E))$ and adapting with the use of assumptions (A.3), (A.4) the proof of Proposition 2.1 and Corollary 2.1 in [3], we have

$$\prod\nolimits_m^{\vartheta_m}(\eta_m, F) \to \prod\nolimits^{\vartheta}(\eta, F) \tag{3.24}$$

where, we recall, $\prod^\vartheta(\eta, \cdot)$ is the transition kernel of (π_i^ϑ); analogously for $\prod_m^{\vartheta_m}(\eta_m, \cdot)$.

Notice now that the convergence (3.24) is equivalent to

$$\pi_1^{m,\vartheta_m} \Rightarrow \pi_1^{\vartheta} \tag{3.25}$$

which proves (3.15) for $n = 0$.

Assume next, (3.15) holds for $n-1$. For $F_1, \ldots, F_{n+1} \in C(P(E))$ we may write

$$\begin{aligned} &E_{\eta_m}\{F_1(\pi_1^{m,\vartheta_m}) \ldots F_n(\pi_n^{m,\vartheta_m}) F_{n+1}(\pi_{n+1}^{m,\vartheta_m})\} = \\ &= E_{\eta_m}\{F_1(\pi_1^{m,\vartheta_m}) \ldots F_n(\pi_n^{m,\vartheta_m}) \prod\nolimits_m^{\vartheta_m}(\pi_n^{m,\vartheta_m}, F_{n+1})\} \end{aligned} \tag{3.26}$$

Furthermore, provided $\pi_n^{m,\vartheta_m}(K_m) \to \pi_n^{\vartheta}(K) \neq \gamma$ we can again adapt Corollary 2.1 from [3] to obtain

$$\prod\nolimits_m^{\vartheta_m}(\pi_n^{m,\vartheta_m}, F_{n+1}) \to \prod\nolimits^{\vartheta}(\pi_n^{\vartheta}, F) \tag{3.27}$$

Finally, using again Lemma 3.1 we have

$$P_\eta\{\pi_n^{\vartheta}(\partial K) > 0\} + P_\eta\{\pi_n^{\vartheta}(K) = \gamma\} = 0 \tag{3.28}$$

At this point we can again use Theorem I.5.5 in [1], obtaining on the basis of the induction hypothesis and the fact that $F_i \in C(P(E))$ $(i = 1, 2, \ldots, n+1)$

$$\begin{aligned} &E_{\eta_m}\{F_1(\pi_1^{m,\vartheta_m}) \ldots F_n(\pi_n^{m,\vartheta_m}) \prod\nolimits_m^{\vartheta_m}(\pi_n^{m,\vartheta_m}, F_{n+1})\} \\ &\to E_\eta\{F_1(\pi_1^{\vartheta}) \ldots F_n(\pi_n^{\vartheta}) \prod\nolimits^{\vartheta}(\pi_n^{\vartheta}, F_{n+1})\} = \\ &= E_\eta\{F_1(\pi_1^{\vartheta}) \ldots F_n(\pi_n^{\vartheta}) F_{n+1}(\pi_{n+1}^{\vartheta})\} = \end{aligned} \tag{3.29}$$

■

Remark. If (see the Remark in Section 1.a) one considers strategies of the form $\vartheta = (u, \psi, \gamma, \eta)$ with $\psi \in C(E)$ having compact support and satisfying $0 \leq \psi \leq 1$, then as approximations one may take $\vartheta_m = (u_m, \psi_m, \gamma_m, \eta_m)$ with u_m, γ_m, η_m as before and $\psi_m \in B(E)$ with compact support satisfying $0 \leq \psi_m \leq 1$ and such that

$$\sup_{x \in E} |\psi_m(x) - \psi(x)| \to 0 \text{ for } m \to \infty$$

In this case the proof of Theorem 3.1 is simpler and does not require assumption (A5). In fact, Lemma 3.1 is not needed any more, since, if $\nu_i^m \Rightarrow \nu_i$ $(i = 1, 2, \ldots, n+1)$, then

$$\begin{aligned} &|\nu_i^m(\psi_m) - \nu_i(\psi)| \leq |\nu_i^m(\psi_m) - \nu_i(\psi)| + |\nu_i^m(\psi_m) - \nu_i(\psi)| \\ &\leq \sup_{x \in E} |\psi_m(x) - \psi(x)| + |\nu_i^m(\psi) - \nu_i(\psi)| \xrightarrow{m \to \infty} 0 \end{aligned}$$

This implies that, instead of (3.16), we have more simply

$$\Gamma \subset \{(\nu_1, \ldots, \nu_{n+1}), \text{ for some } i = 1, 2, \ldots, n, \ \nu_i(\psi) = \gamma\} = \tilde{\Gamma}$$

At this point (3.23) becomes just

$$P_\eta\{\pi_1^\vartheta, \ldots, \pi_{n+1}^\vartheta) \in \tilde{\Gamma}\} \leq \sum_{i=1}^{n} P_\eta\{\pi_i^\vartheta(\psi) = \gamma\}$$

where the quantity on the right is zero simply on the basis of a suitably adapted version of Lemma 2.4 in [3], without the need to assume (A5).

4. COMMENTS ON THE APPLICATIONS FOR THE CONSTRUCTION OF NEARLY OPTIMAL CONTROLS

The results obtained so far concern uniqueness, representation and convergence of invariant measures, which are especially useful for approximation purposes, in particular for the construction of nearly optimal controls. In [3] a full approximation approach has been worked out in the setting and under the assumptions to obtain uniqueness of the invariant measure that were considered there. This approach, that leads to the construction of nearly optimal controls can also be adapted to our setting with our assumptions and, as in [3], consists of three basic steps:

1. approximation of admissible controls,
2. approximation of the original state space E,
3. approximation of the filter process.

In what follows we only briefly sketch the essential aspects of the three steps where Theorem 3.1 is a crucial tool, the details can be obtained by analogy to [3].

Before coming to the description of the three steps we point out that as in [3] steps 1 and 2 can be used for the construction of nearly optimal control functions, which, when applied to the true filter values, yield nearly optimal controls for the original problem. Since the true filter values, that are elements of the space $P(E)$ of measures on E cannot be computed in practice, in step 3 a computable approximate filter is considered. It can be shown that, when applying the nearly optimal control functions resulting from steps 1 and 2 to the approximate filter values, defined in step 3, one still obtains nearly optimal controls.

Let now V denote the set of our control strategies given by

$$\begin{aligned} &V = \{\vartheta = (u, K, \gamma, \eta) \mid u \in C(P(E)), \\ &K \subset E \text{ and is compact}, \ \gamma \in (0,1), \ \eta \in H \subset P(E)\} \end{aligned} \tag{4.1}$$

In step 1, instead of considering the entire set of continuous controls u corresponding to V a subclass is considered, that is defined as follows. Given $L > 0$ as well as a positive integer n, let

$$A(L,n) = \{u \in A \mid u(\nu) = \bar{u}(\nu(\varphi_1), \ldots, \nu(\varphi_n))\} \tag{4.2}$$

where (see (1.3)), $A = C(P(E), U)$, $\bar{u}: R^n \to R$ is Lipschitz with constant L and $\varphi_1, \ldots, \varphi_n, \ldots$ is a dense sequence in $C_0(E)$. The subclass is thus determined by functions defined on a finite dimensional space and its elements approximate the continuous controls corresponding to elements in V. Besides considering approximations to the continuous controls in V one may also consider approximations K_m, γ_m, η_m $(m = 1, 2, \ldots)$ of the remaining elements in the generic strategy ϑ. Let then

$$V(L,n,m) := \{\vartheta = (u, K_m, \gamma_m, \eta_m) \mid u \in A(L,n)\} \tag{4.3}$$

Based on Theorem 3.1, the following result can be proved

Corollary 4.1. *For a suitable choice of K_m, γ_m, η_m we have, for all $\mu \in P(E)$,*

$$\lim_{L,n,m \to \infty} \inf_{\vartheta \in V(L,n,m)} J_\mu(\vartheta) = \inf_{\vartheta \in V} J_\mu(\vartheta)$$

■

Coming to the second step, notice that the original filter process (π_t^ϑ) takes values in the space $P(E)$ of measures on E. Again, to obtain useful approximations, it is desirable to have a filter process taking values in a finite dimensional space. For this purpose the original state space E may be partitioned into a finite number of sets, which can be considered as elements of a new finite state space, implying that the filter process corresponding to an approximation h_m of the observation function $h(x)$, that is constant on the sets of partition, takes values in a simplex. Also functions c, d in the cost functional (1.6) may be approximated by functions c_m, d_m which are constant on the sets of partition. Theorem 3.1 again turns out to be a crucial tool to prove convergence of the cost functional when the partition becomes finer and finer and the step functions h_m, c_m, d_m converges to h, c, d respectively. For the proof of this convergence it is important to consider controls in the approximating class $A(L,n)$, since they are determined by functions acting on finite dimensional vectors.

The computable approximate filter in step 3 is of the form of the filter process of the previous step 2 that takes values in a simplex. While the filter in step 2 served only as a tool to construct a nearly optimal control function, here we want it to be driven by the real observations corresponding to the original model. As a consequence, it is no longer Markov and can not be interpreted as a conditional distribution. It turns out, however, that the pair given by the real filter and the approximate filter mentioned above form a Markov process. Again a suitable version of Theorem 3.1 can be used to show that in the limit, the behaviour of the approximate filter is close to that of the real filter.

Although the three steps, described synthetically above, parallel the procedure in [3], the last step is much simpler in our setting: this is due to the regenerative structure of the filtering processes, implied by the periodic restarting of the original state process from the same measure η.

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